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Participation and Demand Levels for a Joint Project

Ryusuke Shinohara

Faculty of Economics
Shinshu University
Matsumoto 390-8621 Japan
Phone: +81-263-35-4600
Fax: +81-263-37-2344

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Ryusuke Shinohara*
Faculty of Economics, Shinshu University

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Abstract

We examine a voluntary participation problem in public good provision when each agent has a demand level for a public good. The demand level of an agent for a public good is the minimum level of the public good from which she can receive a positive benefit. We show that in the voluntary participation game, the efficient level of the public good is provided at a subgame perfect Nash equilibrium. We also show that there is a subgame perfect Nash equilibrium with efficient provision of the public good that is robust against coordination, as modeled through a strong perfect equilibrium introduced by Rubinstein (1980), and only the efficient subgame perfect Nash equilibrium is supported at the strong perfect equilibrium if every agent has only one demand level. If every agent has more than one threshold, then only the inefficient allocation may be attained at subgame perfect Nash equilibria of the voluntary participation game.

Keywords: Public good; Participation; Demand level; Threshold.
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* Correspondence to: Ryusuke Shinohara. Mailing address: 3-1-1, Asahi, Matsumoto, Nagano, 390-8621, Japan. Tel.:+81-263-37-2951 Fax:+81-263-37-2344. *E-mail address*: ryushinohara@yahoo.co.jp

1 Introduction

We examine a voluntary participation problem in (pure) public good provision when agents want different "at least" levels of the public good. A typical example of the problem is as follows: consider a situation in which there is a metropolis and three cities (A, B, C). City A is nearest to the metropolis, B is the second nearest, and C is the farthest. The mayors of these three cities want to connect their cities with the metropolis by roads, and plan to construct roads to the metropolis jointly. They are assumed to be able to construct roads in the following three segments: from the metropolis to city A (segment 1), from city A to city B (segment 2), and from city B to city C (segment 3). The mayor of each city is satisfied only if her city is connected to the metropolis by roads. Thus, the mayor of cities A, B, and C are satisfied only if roads are built in, at least, segment 1, segments 1 and 2, and all segments, respectively. They all demand different "at least" lengths of the roads. The problem we consider is whether the roads are constructed efficiently in a situation in which they are public goods and participation in the joint project is not coercive. If the roads are public goods, a city can benefit from them at no cost by not participating if others build them at a level with which the city is satisfied. Therefore, there is the possibility that some city does not participate, so that the roads are set inefficiently.

Several researchers have studied the voluntary participation problem and have pointed out the seriousness of the problem by analyzing a **voluntary participation game**. This game consists of two stages: in the first stage, all agents decide whether to participate in the joint production of the public good. In the second stage, knowing who participates, the participants decide the level of the public good and distribute the cost of the public good according to some non-cooperative game (e.g. efficient public good mechanisms such as those of Groves and Ledyard (1977) and Walker (1981) or the contribution game of Bergstrom et al (1986)). If an agent chooses not to participate, she can free-ride. Saijo and Yamato (1999) study the participation game when there is one perfectly divisible public good and one perfectly divisible private good and all agents have the same Cobb-Douglas utility function and the same initial endowment of the private good. They prove that the participation of all agents is less likely to be supported at an equilibrium as the number of agents in the economy increases. As a result, the equilibrium allocation is seldom Pareto-efficient.[1] The same applies to the case in which agents have other preference relations as Healy (2010), Furusawa and Konishi (2011), and Konishi and Shinohara (2012) show.[2]

These studies assume that agents have continuous and monotone preferences. Hence, the benefit of agents from the public good increases continuously and monotonically. However, in the real world, as in the previous example of building roads, there may be a situation in which each agent has a demand level for the public good and she can only enjoy a large benefit if her demand level is fulfilled: the benefit of agents increases discontinuously at the level she wants. The cooperative construction of an irrigation ditch has a structure similar to that of road construction. In the global warming problem, the benefit to each country from the abatement of greenhouse gases may increase discontinuously at some level of abatement. Environment-conscious countries want to reduce greenhouse gasses substantially worldwide, and their demand levels for the abatement are very large.[3] In contrast, countries that are less concerned about global warming are satisfied with a small reduction of greenhouse gasses. By investigating how such a discontinuous jump in benefits affects participation behavior, we reexamine the seriousness of the voluntary participation problem from the viewpoint of allocative efficiency.

The benefit structure we introduce is as follows: let i be an agent. Let $Y_i > 0$ be the threshold level of the public good such that i gains $\theta_i > 0$ if and only if Y_i or more units of

[1] Saijo and Yamato (2010) extend their result to a case in which agents have asymmetric Cobb-Douglas preferences.

[2] Shinohara (2009) obtains a similar negative result in the case in which a public good is discrete. He shows that if the level of the public good is provided in integer units and the participation of many agents is needed for the efficient provision of the public good, the equilibrium level of the public good is inefficient.

[3] Island countries such as the Republic of Maldives and Tuvalu are examples of countries with have high demand levels for the reduction of greenhouse gasses since they are vulnerable to the rise in sea levels caused by global warming and do not benefit from a small abatement.

the public good are produced. We call Y_i the **demand level** of i.

In this paper, we assume that the participants play the (simultaneous) contribution game of Branzei et al (2005) in the second stage of the voluntary participation game. One of the merits of the contribution game is that the allocation that maximizes the total surplus of the participants is supported at a Nash equilibrium of the game. We call the allocation maximizing the total surplus of a set of participants a **group efficient** (*GE*) allocation for the set. The *GE* allocation for the whole set of agents is a Pareto efficient allocation. Thus, by setting the Branzei et al. (2005) game as the second stage, we can guarantee that some set of participants (for example, the whole set of agents) can produce the Pareto efficient allocation in the voluntary participation game. [4]

We first analyze the subgame perfect Nash equilibrium of the voluntary participation game. We first show that for each set of participants, there is a Nash equilibrium in the second-stage game that supports the allocation satisfying *GE*, individual rationality (*IR*), and positive cost share (*PCS*) for the set of participants.[5] As a backward induction hypothesis, we assume that each set of participants selects a Nash equilibrium satisfying these three conditions. Next, we investigate the first-stage game induced by the induction hypothesis. We show that in the induced first-stage game, there is a Nash equilibrium that supports the Pareto efficient allocation. The set of participants supported at the Nash equilibrium is inductively constructed so that if each participant chooses not to participate, her demand level is not fulfilled. In conclusion, in the voluntary participation game, under the benefit structure, there is a subgame perfect Nash equilibrium that achieves Pareto efficiency. We also point out that the public good may be provided Pareto inefficiently at a subgame perfect Nash equilibrium.

Second, we ask whether the possibility of coordination among agents resolves the multiplicity

[4] Trivially, if no set of participants can achieve Pareto efficiency, the voluntary participation game, even in our case, has no subgame perfect Nash equilibrium that produces the Pareto efficient allocation. Saijo and Yamato (1999, 2010) also impose a similar condition on the second-stage game since they set the mechanism that implements the Lindahl allocation as the second stage game. Clearly, in their game, the mechanism provides the Lindahl allocation for each set of participants.

[5] See Definition 1 for the precise definitions of *IR* and *PCS* for a set of participants.

of equilibria. We examine a **strong perfect equilibrium** introduced by Rubinstein (1980). The strong perfect equilibrium is an equilibrium concept for extensive-form games and is immune to unilateral deviation as well as coalitional deviations. Therefore, each strong perfect equilibrium is a subgame perfect Nash equilibrium, but the converse is not true. We show that in the voluntary participation game, (i) a strong perfect equilibrium exists and (ii) only the subgame perfect Nash equilibrium that produces the Pareto efficient allocation is strongly perfect. Thus, coordination modeled through the strong perfect equilibrium leads only to the efficient provision of the public good.

Finally, as an extension of the analysis above, we investigate the case in which the benefit of each agent has jumps at multiple thresholds. We provide examples and show that the voluntary participation game does not necessarily have a subgame perfect Nash equilibrium that produces the Pareto efficient allocation.

Our results imply that whether the allocative efficiency is achieved at an equilibrium depends on the number of thresholds that an agent has. As is shown above, when each agent has only one demand level, the efficient provision of the public good is attained at a subgame perfect Nash equilibrium and only the subgame perfect Nash equilibrium with the Pareto efficient allocation can be achieved through coordination. Therefore, in this case, the voluntary participation problem is not as serious as earlier studies report. However, if an agent has more than one threshold, then the result seems to be similar to the results of the earlier studies: that is, only the inefficient allocation is attained at equilibria.

Related literature other than Saijo and Yamato (1999) and their followers

The voluntary participation game with a public project is also related to this paper. This game is the same as that of Saijo and Yamato (1999), but in the game with a public project, the level of the public good is binary: it is a positive and fixed level (for example one, or zero). Palfrey and Rosenthal (1984), Dixit and Oslon (2000), and Shinohara (2007, 2009) examine this game. Their models are slightly different, but they have a common feature: the positive level of the public good is produced if and only if a certain number of agents participate in public good provision. All of their games have an equilibrium at which the public good is provided Pareto efficiently. Our model substantially generalizes their games. The game of Shinohara (2007, 2009) corresponds to our game in the case in which every agent has the same demand level. In addition, if every agent receives the same benefit from the public good, our game is the same as the games of Palfrey and Rosenthal (1984) and Dixit and Olson (2000). The existence of efficient equilibria is also generalized by our research.

The airport game, which is a classical game in cooperative game theory (Littlechild and Owen, 1973), is relevant to this paper. This game examines the method of cost sharing in maintaining a fixed-length runway among airlines. Each airline demands a different length of runway; for example, one airline may want a half-length of runway, while another may need the full length. The Baker-Thompson rule is the typical of cost sharing, and many studies have characterized it by using the core, the Shapley value, and the nucleolus. Moulin (1994) examines an incentive aspect of cost sharing for the Baker-Thompson rule. Branzei et al (2005) examine an *enterprise game*, which is an extension of the airport game in the sense that the length of the runway is a player choice variable. The airport problem may appear to treat a situation similar to ours, since players have demand levels. However, our game is basically different because in the airport game, non-contributors are excluded from the usage of runways, while in our setting, such an exclusion is impossible. With the exclusion assumed in the airport game, it is easy to induce each agent to participate in sharing the cost of the runway. In fact, Branzei et al. (2005) show the existence of a core in the enterprise game. The core is individually rational and no group of players can block the core. Thus, all agents voluntarily participate in the same (excludable) project and no group of agents deviates and starts another new project. The reason we need examine the participation problem is because of the non-excludability of a public good, as mentioned before.

Branzei et al (2005) provide a simple non-cooperative contribution mechanism that implements the core of the enterprise game at strong Nash equilibria. Thus, their mechanism shows a good performance in implementing an excludable project. Our results show that it also works well in a project *without exclusion*. In the voluntary participation game, agents can commit to free-ride the public good by not participating. Even in such a situation, sufficiently many agents voluntarily participate in the contribution game of Branzei et al (2005), and participants produce the public good Pareto efficiently at a subgame perfect Nash equilibrium of the voluntary participation game. Therefore, we conclude that the mechanism of Branzei et al (2005) achieves a Pareto efficient allocation irrespective of whether exclusion is possible.

The paper is organized as follows: in Sections 2 and 3, we introduce the model and equilibrium concepts. In Section 4, we examine the equilibrium outcomes of the contribution game. In Section 5, we provide the basic properties of the voluntary participation game. In Section 6, we provide an analysis of the subgame perfect Nash equilibrium; in Section 7, we provide an analysis of the strong perfect Nash equilibrium. In Section 8, we provide an extension. Section 9 concludes the paper.

2 The model

Consider an economy in which there is one private and one public good. The level of the public good can be any non-negative real number. For each $y \geq 0$, $c(y)$ is the amount of the private good required to produce y units of the public good. We assume that $c(\cdot)$ is an increasing function and $c(0) = 0$.

Let $N = \{1, 2, \ldots, n\}$ with $n \geq 2$ be the set of agents. When y and $x_i \geq 0$ represent the level of the public good and the contribution to public good production from agent $i \in N$, respectively, agent i's payoff is given by $V_i(y, x_i) = B_i(y) - x_i$. Function $B_i(\cdot)$ is a benefit function of $i \in N$ and is as follows:

$$B_i(y) = \begin{cases} \theta_i > 0 & \text{if } y \geq Y_i \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Agent $i \in N$ has a demand level for the public good, which is denoted by $Y_i > 0$. Equation (1) means that i's benefit from y units of the public good is $\theta_i > 0$ if her demand is fulfilled, and zero otherwise.

The set of demand levels is denoted by $\overline{\mathcal{Y}} \equiv \{\overline{y}^1, \ldots, \overline{y}^t\}$ with $1 \leq t \leq n$. Without loss of generality, suppose that $\overline{y}^1 < \cdots < \overline{y}^t$. Define $N(y) \equiv \{i \in N | Y_i = y\}$ for each $y \in \overline{\mathcal{Y}}$. Then, $\{N(y)\}_{y \in \overline{\mathcal{Y}}}$ is a partition of N. Let $\mathcal{N}(y', y''] \equiv \bigcup_{y:\, y' < y \leq y''} N(y)$ for each pair y', $y'' \in \{0\} \cup \overline{\mathcal{Y}}$ such that $y' < y''$.

Let $(y^*, (x_i^*)_{i \in N}) \in [\{0\} \cup \overline{\mathcal{Y}}] \times \mathbb{R}_+^n$ denote a (Pareto) efficient allocation. Total surplus $\sum_{i \in N} B_i(y) - c(y)$ is maximized at y^* and $\sum_{j \in N} x_j^* = c(y^*)$. We assume that

$$\sum_{i \in N} B_i(y) - c(y) > 0 \text{ for some } y \in \overline{\mathcal{Y}}, \tag{2}$$

so that $y^* > 0$ and $\sum_{i \in N} B_i(y^*) - c(y^*) > 0$.

We consider a situation in which there exists an opportunity for the joint production of the public good. Each agent can decide whether to participate in the production. We consider the following two-stage game. In the first stage, agents simultaneously decide whether to participate (participation stage). In the second stage, knowing the other agents' participation decisions, the agents who chose to participate jointly produce the public good and distribute its cost (public good provision stage). The agents who do not participate can free-ride the public good.

The participation stage

In the first stage, each agent chooses 0 (not participating) or 1 (participating). Let $\{0, 1\}$ be the set of actions for each agent in the first stage.

The contribution stage

We model the second-stage game as the contribution game of Branzei et al. (2005). The Branzei et al. game is a simultaneous game with complete information. Each participant chooses a vector of marginal contributions to each possible increase of the public good provision from 0 to $\overline{y}^1$, from $\overline{y}^1$ to $\overline{y}^2$, ..., and from $\overline{y}^{t-1}$ to $\overline{y}^t$. Let $P \subseteq N$ be a set of participants. For each $\overline{y}^l \in \overline{\mathcal{Y}}$ and each $i \in P$, let $x_i(\overline{y}^l) \in \mathbb{R}_+$ be a marginal contribution of i to the increase from $\overline{y}^{l-1}$ to $\overline{y}^l$. Denote $\boldsymbol{x_i} = (x_i(y))_{y \in \overline{\mathcal{Y}}} \in \mathbb{R}_+^t$ by a marginal contribution vector announced by $i \in P$. Let $\boldsymbol{x_P} = (\boldsymbol{x_i})_{i \in P} \in \mathbb{R}_+^{t|P|}$.

The level of the public good is determined as follows: the public good of $\overline{y}^l \in \overline{\mathcal{Y}}$ units is provided as long as the aggregate marginal contribution $\sum_{i \in P} x_i(\overline{y}^l)$ covers the incremental cost $c(\overline{y}^l) - c(\overline{y}^{l-1})$. If $\sum_{i \in P} x_i(\overline{y}^k) < c(\overline{y}^k) - c(\overline{y}^{k-1})$ for some $\overline{y}^k \in \overline{\mathcal{Y}}$, then the incremental provision from $\overline{y}^{k-1}$ to $\overline{y}^k$ is not realized and also not the higher incremental provision. Formally, given $\boldsymbol{x_P} \in \mathbb{R}_+^{t|P|}$, the level of the public good is given by

$$\psi(\boldsymbol{x_P}) \equiv \max \left\{ \overline{y}^l \in \overline{\mathcal{Y}} | \sum_{i \in P} x_i(\overline{y}^k) \geq c(\overline{y}^k) - c(\overline{y}^{k-1}) \text{ for each } \overline{y}^k \text{ such that } \overline{y}^k \leq \overline{y}^l \right\}.$$

Participants never get money back, whether the aggregate marginal contribution to an incremental provision is insufficient or exceeds the incremental cost. Participant i's payoff at $\boldsymbol{x_P}$ is $V_i\left(\psi(\boldsymbol{x_P}), \sum_{y \in \overline{\mathcal{Y}}} x_i(y)\right) = B_i(\psi(\boldsymbol{x_P})) - \sum_{y \in \overline{\mathcal{Y}}} x_i(y)$. For each $i \in P$ and each $\boldsymbol{x_P} \in \mathbb{R}_+^{t|P|}$, denote $\pi_i(\boldsymbol{x_P}) \equiv V_i\left(\psi(\boldsymbol{x_P}), \sum_{y \in \overline{\mathcal{Y}}} x_i(y)\right)$. Even if a participant contributes nothing, she benefits from the public good if her demand is fulfilled. The formulation of the second stage is common to every set of participants.

Strategies

The set of strategies of $i \in N$, denoted by $\mathcal{S}_i$, is $\mathcal{S}_i \equiv \{0,1\} \times \{\gamma_i : \{P \subseteq N | \, i \in P\} \to \mathbb{R}_+^t\}$. Set $\{0,1\}$ is the set of first stage actions. What i announces in the second stage depends on who participates in the contribution game. Hence, the second-stage strategy is a function that corresponds a marginal contribution vector with a set of participants. We define $S_i^1 \equiv \{0,1\}$ and $S_i^2 \equiv \{\gamma_i : \{P \subseteq N | \, i \in P\} \to \mathbb{R}_+^t\}$. We denote a typical element in $\mathcal{S}_i$ by $s_i = (s_i^1, \gamma_i)$ such that $s_i^1 \in S_i^1$ and $\gamma_i \in S_i^2$. We denote $\mathcal{S} \equiv \prod_{i \in N} \mathcal{S}_i$, $S^j \equiv \prod_{i \in N} S_i^j$ for each $j \in \{1,2\}$. For each $P \subseteq N$, each $i \in P$, each z such that $\overline{y}^z \in \overline{\mathcal{Y}}$, and each $\gamma_i \in S_i^2$, $\gamma_i^z(P) \in \mathbb{R}_+$ represents a marginal contribution from i to the increase of the public good from $\overline{y}^{z-1}$ to $\overline{y}^z$ when P is a set of participants.

Payoffs

Let $s \in \mathcal{S}$ be a strategy profile. Denote $P(s) \equiv \{i \in N | \, s_i^1 = 1\}$ (the set of participants at s). At s, $i \in N$ obtains $U_i(s) \equiv V_i(\psi((\gamma_i(P(s)))_{i \in P}), \sum_{z=1}^t \gamma_i^z(P(s)))$ if $i \in P(s)$ and

$U_i(s) \equiv V_i(\psi((\gamma_i(P(s)))_{i \in P}), 0)$ otherwise.

The voluntary participation game is a list $\Gamma = [N, \mathcal{S}, (U_i)_{i \in N}]$.

3 Equilibrium concepts

We adopt two equilibrium concepts. One is a **subgame perfect Nash equilibrium**, which is the standard equilibrium notion for multi-stage games; the other is a **strong perfect equilibrium**, which is introduced by Rubinstein (1980). The strong perfect equilibrium is an extension of a strong Nash equilibrium (Aumann, 1959) to multi-stage games and is immune to all possible coalitional deviations.

A strategy profile $s \in \mathcal{S}$ is a **subgame perfect Nash equilibrium** if s assigns a Nash equilibrium for each subgame: $(\gamma_i(P))_{i \in P}$ is a Nash equilibrium of the contribution game for each $P \subseteq N$ and s is a Nash equilibrium of the whole game. We call a subgame perfect Nash equilibrium that provides the efficient (inefficient) level of a public good on the equilibrium path an **efficient** (**inefficient**, respectively) **subgame perfect Nash equilibrium**.

For each $D \subseteq N$, let $-D = N \backslash D$. For each $D \subseteq N$, $\boldsymbol{x_D} \in \mathbb{R}_+^{t|D|}$ denotes a profile of the marginal contribution vectors of D and $s_D \in \prod_{i \in D} \mathcal{S}_i$ denotes a profile of the strategies of D. For notational simplicity, denote $\boldsymbol{x} = \boldsymbol{x_N} \in \mathbb{R}_+^{tn}$ and $s = s_N \in \mathcal{S}$.

The strong Nash equilibrium is a Nash equilibrium that is immune to all coalitional deviations. Let $P \subseteq N$. A marginal contribution vector $\boldsymbol{x_P} \in \mathbb{R}_+^{t|P|}$ is a strong Nash equilibrium of the contribution game when P is the set of participants if there are no $D \subseteq P$ and $\boldsymbol{x'_D} \in \mathbb{R}_+^{t|D|}$ such that $\pi_i(\boldsymbol{x'_D}, \boldsymbol{x_{P \backslash D}}) > \pi_i(\boldsymbol{x_P})$ for each $i \in D$.

We next define the strong perfect equilibrium. Profile $s \in \mathcal{S}$ is a strong Nash equilibrium of Γ if there are no $D \subseteq N$ and $s'_D \in \prod_{i \in D} \mathcal{S}_i$ such that $U_i(s'_D, s_{-D}) > U_i(s)$ for each $i \in D$. Profile $s \in \mathcal{S}$ is a **strong perfect equilibrium** of Γ if it assigns a strong Nash equilibrium for each subgame: $(\gamma_i(P))_{i \in P}$ is a strong Nash equilibrium of the contribution game for each $P \subseteq N$ and s is a strong Nash equilibrium of the whole game.

The strong perfect equilibrium is stable against coordination among agents within and across stages. Clearly, every strong perfect equilibrium is a subgame perfect Nash equilibrium and the converse is not true.

We focus on the case in which agents use only the pure strategies.

4 Equilibrium analysis of the second stage

The contribution game has many Nash equilibria. Some equilibria are efficient with respect to the participants' preferences, while others are not. We investigate a strong Nash equilibrium.

Lemma 1 presents a necessary condition for a Nash equilibrium of the second-stage game.

Lemma 1 Let $P \subseteq N$. Let $\boldsymbol{x_P} \in \mathbb{R}_+^{t|P|}$ be a marginal contribution vector at a Nash equilibrium of the contribution game. Then, (i) $x_i(y) = 0$ for each $i \in P$ such that $Y_i > \psi(\boldsymbol{x_P})$ and each $y \in \overline{\mathcal{Y}}$, (ii) $x_i(y) = 0$ for each $i \in P \cap \mathcal{N}(0, \psi(\boldsymbol{x_P})]$ and each $y \in \overline{\mathcal{Y}}$ such that $y > Y_i$, and (iii) $\psi(\boldsymbol{x_P}) \leq \overline{y}$, where $\overline{y} \equiv \max \left\{ \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P} B_i(y) - c(y) \right\}$.

Proof. (i) Suppose that $x_i(y) > 0$ for some $i \in P$ such that $Y_i > \psi(\boldsymbol{x_P})$ and some $y \in \overline{\mathcal{Y}}$. Then, i obtains the payoff $-\sum_{z=1}^{t} x_i(\overline{y}^z) < 0$. If she reduces her contribution, then she is made better off.

(ii) Suppose that $x_i(\overline{y}^l) > 0$ for some $i \in P \cap \mathcal{N}(0, \psi(\boldsymbol{x_P})]$ and some $\overline{y}^l \in \overline{\mathcal{Y}}$ such that $\overline{y}^l > Y_i$. Let $\boldsymbol{x'_i} = (x'_i(y))_{y \in \overline{\mathcal{Y}}}$ be such that $x'_i(\overline{y}^l) = 0$ and $x'_i(y) = x_i(y)$ for each $y \in \overline{\mathcal{Y}} \backslash \{\overline{y}^l\}$. We first consider the case of $\psi(\boldsymbol{x_P}) < \overline{y}^l$. In this case, by the definition of ψ, $\psi(\boldsymbol{x_P}) = \psi(\boldsymbol{x'_i}, \boldsymbol{x_{P\backslash\{i\}}})$. Thus, $\pi_i(\boldsymbol{x_P}) = \theta_i - \sum_{z=1}^{t} x_i(\overline{y}^z) < \theta_i - \sum_{z=1}^{t} x'_i(\overline{y}^z) = \pi_i(\boldsymbol{x'_i}, \boldsymbol{x_{P\backslash\{i\}}})$, which implies that $\boldsymbol{x_P}$ is not a Nash equilibrium. We also consider the case of $\psi(\boldsymbol{x_P}) \geq \overline{y}^l$. If i switches from $\boldsymbol{x_i}$ to $\boldsymbol{x'_i}$, $\overline{y}^l$ may not be provided. By the construction of $\boldsymbol{x'_i}$ and the definition of ψ, $\psi(\boldsymbol{x'_i}, \boldsymbol{x_{P\backslash\{i\}}}) \geq \overline{y}^{l-1}$. Since $\overline{y}^l > Y_i$ and $\overline{\mathcal{Y}}$ is discrete, then $\overline{y}^{l-1} \geq Y_i$. Then, $\psi(\boldsymbol{x'_i}, \boldsymbol{x_{P\backslash\{i\}}}) \geq Y_i$. Thus, i can switch from $\boldsymbol{x_i}$ to $\boldsymbol{x'_i}$ in such a way that her demand level is fulfilled and her contribution declines.

(iii) Suppose that $\psi(\boldsymbol{x_P}) > \overline{y}$. Clearly, $\psi(\boldsymbol{x_P}) \notin \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P} B_i(y) - c(y)$. Then, $\sum_{j \in P \cap \mathcal{N}(\overline{y}, \psi(\boldsymbol{x_P})]} \theta_j < c(\psi(\boldsymbol{x_P})) - c(\overline{y})$. By (i) and (ii), agents in $P \cap \mathcal{N}(\overline{y}, \psi(\boldsymbol{x_P})]$ contribute

$c(\psi(\boldsymbol{x_P})) - c(\overline{y})$. They may also contribute the provision of a lower level of the public good. Thus, $\sum_{j \in P \cap \mathcal{N}(\overline{y}, \psi(\boldsymbol{x_P})]} \sum_{y \in \overline{\mathcal{Y}}} x_j(y) \geq c(\psi(\boldsymbol{x_P})) - c(\overline{y})$, which implies that there exists $i \in P \cap \mathcal{N}(\overline{y}, \psi(\boldsymbol{x_P})]$ such that $\theta_i - \sum_{y \in \overline{\mathcal{Y}}} x_i(y) < 0$. Since $\theta_i > 0$, then $\sum_{y \in \overline{\mathcal{Y}}} x_i(y) > 0$. If an agent's total contribution is zero, then her payoff is at least zero and she is made better off. This contradicts the idea that $\boldsymbol{x_P}$ is a Nash equilibrium. ■

Let $P \subseteq N$. Let $\overline{y}^l \in \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P} B_j(y) - c(y)$. Let us set $\boldsymbol{x_P} \in \mathbb{R}_+^{t|P|}$ as follows:

(a) For each $i \in P$ such that $Y_i > \overline{y}^l$ and each $y \in \overline{\mathcal{Y}}$, $x_i(y) = 0$. For each $i \in P \cap \mathcal{N}(0, \overline{y}^l]$ and each $y \in \overline{\mathcal{Y}}$ such that $y > Y_i$, $x_i(y) = 0$.

(b) For each $k \in \{1, \ldots, l\}$ such that $\overline{y}^k \in \overline{\mathcal{Y}}$,

$$\sum_{j \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]} x_j(\overline{y}^k) = c(\overline{y}^k) - c(\overline{y}^{k-1}) \tag{3}$$

$$\text{and } \theta_i - \sum_{z=k}^{l} x_i(\overline{y}^z) \geq 0 \text{ for each } i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]. \tag{4}$$

(c) For each $i \in P \cap \mathcal{N}(0, \overline{y}^l]$, $\sum_{z=1}^{l} x_i(\overline{y}^z) > 0$.

Property (a) is based on Lemma 1. In (b), (3) means that the budget balance condition holds for each increase of the public good up to $\overline{y}^l$ and (4) means that the payment of each participant does not outweigh her benefit. Property (c) means that each participant whose demand level is fulfilled pays a positive fee.

Lemma 2 shows that we can construct such an $\boldsymbol{x_P}$.

Lemma 2 There is a vector of marginal contributions that satisfies (a), (b), and (c).

Proof. Clearly, we can take a marginal contribution vector that satisfies (a). We show the existence of the vector that satisfies (b) by induction. First, we consider the case of $k = l$. Since $\overline{y}^l$ maximizes $\sum_{j \in P} B_j(y) - c(y)$, then $\sum_{j \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l]} \theta_j \geq c(\overline{y}^l) - c(\overline{y}^{l-1})$. Hence, we can take $(x_i(\overline{y}^l))_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l]}$ that satisfies (3) and (4) for $k = l$. Let $k \leq l - 1$ and we assume as an induction hypothesis that (3) and (4) hold for each number greater than k. We show that (3) and (4) also hold for k. Since $\overline{y}^l$ maximizes $\sum_{j\in P} B_j(y) - c(y)$,

$$\sum_{j\in P\cap\mathcal{N}(\overline{y}^{k-1},\overline{y}^l]} \theta_j \geq c(\overline{y}^l) - c(\overline{y}^{k-1}). \tag{5}$$

By (5),

$$\sum_{j\in P\cap\mathcal{N}(\overline{y}^{k},\overline{y}^l]} \theta_j - (c(\overline{y}^l) - c(\overline{y}^k)) + \sum_{j\in P\cap N(\overline{y}^k)} \theta_j \geq c(\overline{y}^k) - c(\overline{y}^{k-1}).$$

By the induction hypothesis, $c(\overline{y}^l) - c(\overline{y}^k) = \sum_{j\in P\cap\mathcal{N}(\overline{y}^k,\overline{y}^l]} \sum_{z=k+1}^{l} x_j(\overline{y}^z)$. Hence,

$$\sum_{j\in P\cap\mathcal{N}(\overline{y}^{k},\overline{y}^l]} \left[\theta_j - \sum_{z=k+1}^{l} x_j(\overline{y}^z)\right] + \sum_{j\in P\cap N(\overline{y}^k)} \theta_j \geq c(\overline{y}^k) - c(\overline{y}^{k-1}).$$

Since $\theta_j - \sum_{z=k+1}^{l} x_j(\overline{y}^z) \geq 0$ for each $j \in P \cap \mathcal{N}(\overline{y}^k, \overline{y}^l]$, we can take $x_i(\overline{y}^k)$ for each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]$ that satisfies (3) and (4).

If $\sum_{z=1}^{l} x_i(\overline{y}^z) = 0$ for some agent $i \in P \cap \mathcal{N}(0, \overline{y}^l]$, then there is $j \in P \cap \mathcal{N}(0, \overline{y}^l]\backslash\{i\}$ such that $x_j(\overline{y}^z) > 0$ for some $\overline{y}^z \leq Y_i$. If we reduce the contribution of j slightly and add this amount to i's contribution, then i's contribution is positive. Hence, we can construct a profile of marginal contribution vectors in a way that satisfies (c), in addition to (a) and (b). ■

Note that $\overline{y}^l$ units of the public good are provided at this marginal contribution vector.

We introduce several properties of the second-stage outcome.

Definition 1 Let $P \subseteq N$. Let $(\overline{y}, (\sum_{y\in\overline{\mathcal{Y}}} x_i(y))_{i\in P})$ be a second-stage outcome.

(1.1) $(\overline{y}, (\sum_{y\in\overline{\mathcal{Y}}} x_i(y))_{i\in P})$ satisfies **group efficiency** (*GE*) for P if $\overline{y} \in \arg\max_{y\in\overline{\mathcal{Y}}} \sum_{i\in P} B_i(y) - c(y)$ and $\sum_{i\in P}\sum_{y\in\overline{\mathcal{Y}}} x_i(y) = c(\overline{y})$.

(1.2) $(\overline{y}, (\sum_{y\in\overline{\mathcal{Y}}} x_i(y))_{i\in P})$ satisfies **individual rationality** (*IR*) for P if $B_i(\overline{y}) \geq \sum_{y\in\overline{\mathcal{Y}}} x_i(y)$ for each $i \in P$.

(1.3) $(\sum_{y\in\overline{\mathcal{Y}}} x_i(y))_{i\in P}$ satisfies **positive cost share** (*PCS*) for P if $\sum_{y\in\overline{\mathcal{Y}}} x_i(y) > 0$ for each $i \in P \cap \mathcal{N}(0, \overline{y}]$.

GE requires that a set of participants produce the public good in a way that maximizes the aggregate surplus of participants and satisfies budget balance. The outcome that satisfies *GE* is Pareto efficient with respect to the participants' preferences. *IR* requires that the benefit from the public good should not be less than the aggregate contribution. *PCS* requires that each participant whose demand level is fulfilled should pay a positive fee. *IR* and *PCS* imply that for each $P \subseteq N$ and each $i \in P$, i pays a positive fee if and only if i's demand level is fulfilled. The following summarizes the properties of vectors with (a), (b), and (c).

Lemma 3 Let $P \subseteq N$. Let $\boldsymbol{x_P} \in \mathbb{R}_+^{t|P|}$ be a marginal contribution vector that satisfies (a), (b), and (c). Then, the allocation at $\boldsymbol{x_P}$ satisfies *GE*, *IR*, and *PCS*.

A profile of marginal contribution vectors that satisfies (a) and (b) is robust to coalitional deviations.[6]

Lemma 4 For each $P \subseteq N$, $\boldsymbol{x_P}$ satisfying (a) and (b) is a strong Nash equilibrium in the contribution game.

Proof. Trivially, $\boldsymbol{x_P}$ is a Nash equilibrium because no $i \in P$ is made better off by increasing or decreasing her contribution. Suppose, to the contrary, that there is a coalition $D \subseteq P$ and its deviation $\boldsymbol{x'_D} \in \mathbb{R}_+^{t|D|}$ that improve each $i \in D$'s payoff. Denote $\boldsymbol{x'_P} = (\boldsymbol{x'_D}, \boldsymbol{x_{P \setminus D}})$ and $\overline{y}^k = \psi(\boldsymbol{x'_P})$. Since $\sum_{j \in P} B_j(\overline{y}^l) - c(\overline{y}^l) \geq \sum_{j \in P} B_j(y) - c(y)$ for each $y > \overline{y}^l$, then $\sum_{j \in P \cap \mathcal{N}(\overline{y}^l, y]} \theta_j \leq c(y) - c(\overline{y}^l)$. Hence, if $\overline{y}^k > \overline{y}^l$, then no member of D is made better off. If $\overline{y}^k < \overline{y}^l$, then $c(\overline{y}^{k+1}) - c(\overline{y}^k) > \sum_{j \in P} x'_j(\overline{y}^{k+1})$. Recall that $x_i(\overline{y}^{k+1}) > 0$ only if $i \in \mathcal{N}(\overline{y}^k, \overline{y}^l] \cap P$. Hence, $\mathcal{N}(\overline{y}^k, \overline{y}^l] \cap D \neq \emptyset$ and every $i \in \mathcal{N}(\overline{y}^k, \overline{y}^l] \cap D$ obtains $\theta_i - \sum_{z=1}^{l} x_i(\overline{y}^z)$ before the deviation and $-\sum_{z=1}^{l} x'_i(\overline{y}^z)$ after the deviation. By (4), $\theta_i - \sum_{z=1}^{l} x_i(\overline{y}^z) \geq -\sum_{z=1}^{l} x'_i(\overline{y}^z)$, which is a contradiction. ■

[6] Property (c) is not necessary for the marginal contribution vectors to be strong Nash equilibria. However, it saves several steps in the proof of Lemma 8. Without it, we can show the lemma, but the proof is involved.

5 The participation game induced from the second-stage outcome

In the contribution game, multiple strong Nash equilibria exist in general. The payoff allocations at the strong Nash equilibria are also multiple. We assume that each set of participants $P \subseteq N$ chooses one of the strong Nash equilibria whose allocation satisfy *GE*, *IR*, and *PCS* in the corresponding contribution game and that each agent has the same prediction of which strong Nash equilibrium is played.

Let $(\gamma_i)_{i \in N}$ be such that $(\gamma_i(P))_{i \in P}$ is a strong Nash equilibrium that satisfies (a), (b), and (c) in the corresponding contribution game for each $P \subseteq N$. Given $(\gamma_i)_{i \in N}$, the voluntary participation game is reduced to the following game: each agent $i \in N$ chooses 0 or 1, simultaneously. Let $P \subseteq N$ be a set of participants. Each $i \in P$ pays $\sum_{z=1}^{t} \gamma_i^z(P)$ and P provides $\psi((\gamma_i(P))_{i \in P})$ units of the public good. Let $y^P \equiv \psi((\gamma_i(P))_{i \in P})$, $x_i^P \equiv \sum_{z=1}^{t} \gamma_i^z(P)$ for each $i \in P$, and $x_i^P \equiv 0$ for each $i \notin P$. By Lemma 3, $y^P \in \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P} B_i(y) - c(y)$, $\sum_{i \in P} x_i^P = c(y^P)$, $B_i(y^P) \geq x_i^P$ for each $i \in P$, and $x_i^P > 0$ for each $i \in P \cap \mathcal{N}(0, y^P]$.

5.1 A Nash equilibrium set of participants

We characterize a set of participants that is supported at a pure-strategy Nash equilibrium of the induced participation game.

Lemma 5 Given that the second-stage outcome satisfies *GE*, *IR*, and *PCS* for each set of participants, a set of participants $P \subseteq N$ is supported at a Nash equilibrium if and only if

$$\text{for each } i \in P, \text{ if } x_i^P > 0, \text{ then } y^{P \setminus \{i\}} < Y_i, \text{ and} \tag{IS}$$

$$\text{for each } i \notin P, \text{ if } y^{P \cup \{i\}} \geq Y_i > y^P, \text{ then } \theta_i - x_i^{P \cup \{i\}} = 0. \tag{ES}$$

Proof. (Necessity) If $x_i^P > 0$ and $y^{P \setminus \{i\}} \geq Y_i$ for some $i \in P$, then i wants to switch from participating to not participating. If there is $i \notin P$ such that $y^{P \cup \{i\}} \geq Y_i > y^P$ and $\theta_i - x_i^{P \cup \{i\}} > 0$, then agent i is better off participating in P. Hence, for each $i \notin P$, if $y^{P \cup \{i\}} \geq Y_i > y^P$, then $\theta_i - x_i^{P \cup \{i\}} \leq 0$. It is immediate from *IR* that *ES* holds.

(Sufficiency) By *IR*, $V_i(y^P, x_i^P) \geq 0$ for each $i \in P$. By *IS* and *PCS*, $V_i(y^{P \setminus \{i\}}, x_i^{P \setminus \{i\}}) = 0$

for each $i \in P$ such that $x_i^P > 0$. Let $k \in P$ be such that $x_k^P = 0$. By *PCS*, $Y_k > y^P$. Hence, $V_k(y^P, x_i^P) = 0$. Since $y^{P \setminus \{k\}} = y^P < Y_k$, then $V_k(y^{P\setminus\{k\}}, x_k^{P\setminus\{k\}}) = 0$. Thus, k is not made better off by switching from participating to not participating. Let $j \notin P$ be such that $y^{P\cup\{j\}} \geq Y_j$. Then, $V_j(y^{P\cup\{j\}}, x_j^{P\cup\{j\}}) = \theta_j - x_j^{P\cup\{j\}}$. If $y^{P\cup\{j\}} \geq Y_j > y^P$, then by *ES*, $V_j(y^{P\cup\{j\}}, x_j^{P\cup\{j\}}) = V_j(y^P, x_j^P) = 0$. If $y^{P\cup\{j\}} \geq y^P \geq Y_j$, then by *PCS*, $V_j(y^{P\cup\{j\}}, x_j^{P\cup\{j\}}) < V_j(y^P, x_j^P) = \theta_j$. Clearly, no $i \notin P$ such that $y^{P\cup\{i\}} < Y_i$ switches to participate. ■

Condition *IS* means that no member of $P(s)$ gains by switching to not participating. Condition *ES* means that no outsider of $P(s)$ gains by joining in $P(s)$. These are adaptations of the internal and external stability of d'Aspremont et al (1983).

5.2 Properties of possible demand levels

In the induced participation game, under *GE*, if $P \subseteq N$ is a set of participants and P produces $\overline{y}^l \in \overline{\mathcal{Y}}$ units of the public good, then $\sum_{j\in P} B_j(\overline{y}^l) - c(\overline{y}^l) \geq \sum_{j \in P} B_j(y) - c(y)$ for each $y \in \overline{\mathcal{Y}}$. This condition implies that

1. $\displaystyle\sum_{j\in\mathcal{N}(\overline{y}^k,\overline{y}^l]\cap P} \theta_j \geq c(\overline{y}^l) - c(\overline{y}^k)$ for each $\overline{y}^k \in \{0\} \cup \overline{\mathcal{Y}}$ such that $\overline{y}^l > \overline{y}^k$ and
2. $\displaystyle\sum_{j\in\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap P} \theta_j \leq c(\overline{y}^k) - c(\overline{y}^l)$ for each $\overline{y}^k \in \{0\} \cup \overline{\mathcal{Y}}$ such that $\overline{y}^l < \overline{y}^k$.

Given $(\gamma_i)_{i\in N}$, denote $\mathcal{Y} \equiv \{y \in \overline{\mathcal{Y}} | \, y^P = y \text{ for some } P \subseteq N\}$: $y \in \mathcal{Y}$ is the level of the public good that some set of participants produces in the induced participation game. We call a demand level belonging to $\mathcal{Y}$ a **possible demand level**. Let $\mathcal{Y} \equiv \{y^1, \ldots, y^m\}$ be such that $1 \leq m \leq t$ and $y^{l-1} < y^l$ for each $l \in \{2, \ldots, m\}$. Lemma 6 is a basic property of $\mathcal{Y}$.

Lemma 6 Given $(\gamma_i)_{i\in N}$ that satisfies (a), (b), and (c), $\sum_{j\in\mathcal{N}(y^{l-1},y^l]} \theta_j \geq c(y^l) - c(y^{l-1})$ for each $y^l \in \mathcal{Y}$, where $y^0 \equiv 0$.

Proof. Suppose, to the contrary, that $\sum_{j\in\mathcal{N}(y^{l-1},y^l]} \theta_j < c(y^l) - c(y^{l-1})$ for some $y^l \in \mathcal{Y}$. Let $P \subseteq N$ be such that $y^P = y^l$. By *GE*, $\sum_{j\in P\cap\mathcal{N}(y^{l-1},y^l]} \theta_j \geq c(y^l) - c(y^{l-1})$. However,

$\sum_{j \in P \cap \mathcal{N}(y^{l-1}, y^l]} \theta_j \leq \sum_{j \in \mathcal{N}(y^{l-1}, y^l]} \theta_j < c(y^l) - c(y^{l-1})$, which is a contradiction. ■

Note that we do not assume any condition for the preferences and cost except for (2). There may be a case in which $\sum_{j \in N(\overline{y}^l)} \theta_j < c(\overline{y}^l) - c(\overline{y}^{l-1})$ for some $\overline{y}^l \in \overline{\mathcal{Y}}$. Lemma 6 says that if the level of the public good increases from one possible demand level to a higher possible demand level, then the sum of the marginal benefits is not less than the incremental cost.

Lemma 7 Given $(\gamma_i)_{i \in N}$ that satisfies (a), (b), and (c), $\arg\max_{y \in \mathcal{Y}} \sum_{j \in P} B_j(y) - c(y) \subseteq \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P} B_j(y) - c(y)$ for each $P \subseteq N$.

Proof. Suppose, to the contrary, that there exists $y' \in \arg\max_{y \in \mathcal{Y}} \sum_{j \in P} B_j(y) - c(y) \setminus \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P} B_j(y) - c(y)$. Then, there exists $y'' \in \overline{\mathcal{Y}} \setminus \mathcal{Y}$ such that $\sum_{j \in P} B_j(y'') - c(y'') > \sum_{j \in P} B_j(y') - c(y')$. By the construction of $(\gamma_i)_{i \in N}$, $y^P \in \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P} B_j(y) - c(y)$; hence, $\sum_{j \in P} B_j(y^P) - c(y^P) \geq \sum_{j \in P} B_j(y') - c(y')$. Since $y^P \in \mathcal{Y}$ and $y' \in \arg\max_{y \in \mathcal{Y}} \sum_{j \in P} B_j(y) - c(y)$, then $\sum_{j \in P} B_j(y^P) - c(y^P) \leq \sum_{j \in P} B_j(y') - c(y')$. In conclusion, $\sum_{j \in P} B_j(y'') - c(y'') > \sum_{j \in P} B_j(y') - c(y') = \sum_{j \in P} B_j(y^P) - c(y^P)$, which implies that y^P does not maximize $\sum_{j \in P} B_j(y) - c(y)$, a contradiction. ■

By Lemma 7, the efficient level of the public good within $\mathcal{Y}$ is also efficient within $\overline{\mathcal{Y}}$. By Lemma 6, $y^m \in \arg\max_{y \in \mathcal{Y}} \sum_{j \in N} B_j(y) - c(y)$. By Lemma 7, y^m is an efficient level of the public good.

6 Subgame perfect Nash equilibria of the participation game

In this section, we assume that for each $P \subseteq N$, if P is a set of participants, then P chooses one of allocations that satisfy *IR*, *PCS*, and *GE'* for P, defined as follows:

Group efficiency with the maximal public good (*GE'*) for $P \subseteq N$: the second stage outcome when P is a set of participants $(y^P, (x_i^P)_{i \in P})$ satisfies $y^P = \max \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P} B_i(y) - c(y)$ and $\sum_{i \in P} x_i^P = c(y^P)$.

The group efficient level of the public good for a set of participants may be multiple. *GE'* requires that if there are multiple group-efficient levels of the public good for a set of participants, the participants produce the greatest level among them.[7] Hereafter, $\mathcal{Y}$ is the set of possible demand levels under *GE'*. Note also that as in Section 4, we can construct a strong Nash equilibrium that supports an allocation with *GE'*, *IR*, and *PCS*.

6.1 The existence of an efficient subgame perfect Nash equilibrium

Lemma 8 If the second-stage allocation satisfies *GE'* for each $P \subseteq N$, then for each $y^l \in \mathcal{Y}$, there exists a set of participants P such that P satisfies *IS* and $y^P = y^l$.

Proof. We construct an internally stable set of participants that produces y^m units of the public good. We inductively define $\{P^m, P^{m-1}, \ldots, P^1\}$ such that $P^k \subseteq \mathcal{N}(y^{k-1}, y^k]$ for each $y^k \in \mathcal{Y}$ from step m to step 1.

Step $\boldsymbol{m}$. First, pick up P^m such that

$$P^m \in \arg \min_{P \subseteq \mathcal{N}(y^{m-1}, y^m]} \sum_{j \in P} \theta_j \text{ subject to } \sum_{j \in P} \theta_j \geq c(y^m) - c(y^{m-1}).$$

After taking P^m, go to Step $m - 1$.

Step $\boldsymbol{l} \in \{1, \ldots, m-1\}$. Suppose that we have picked up P^m, ..., P^{l+1} along this way. At Step l, we set P^l.

(l.a) If $\sum_{j \in P^m \cup \ldots \cup P^{l+1}} \theta_j < c(y^m) - c(y^{l-1})$, then

$$P^l \in \arg \min_{P \subseteq \mathcal{N}(y^{l-1}, y^l]} \sum_{j \in P} \theta_j \text{ subject to } \sum_{j \in P} \theta_j \geq c(y^m) - c(y^{l-1}) - \sum_{j \in P^m \cup \ldots \cup P^{l+1}} \theta_j.$$

(l.b) Otherwise, $P^l = \emptyset$.

After defining P^l, go to Step $l - 1$. The following steps continue to set P^1 similarly.

Note that we can pick up from P^m to P^1 by Lemma 6. Let $P^* \equiv P^1 \cup \cdots \cup P^m$.

[7] Note that this is just a tie-breaking rule, and we can modify the proof of our results according to this tie-breaking rule.

Claim 1 It follows that $y^{P^*} = y^m$.

Proof of Claim 1. We need to show that $y^m = \max \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P^*} B_j(y) - c(y)$. Since P^* satisfies

$$\sum_{j \in P^m \cup \cdots \cup P^k} \theta_j \geq c(y^m) - c(y^{k-1}) \text{ for each } k \text{ such that } y^k \in \mathcal{Y}$$

where $y^0 \equiv 0$, then $y^m \in \arg\max_{y \in \mathcal{Y}} \sum_{j \in P^*} B_j(y) - c(y)$. By Lemma 7, $y^m \in \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P^*} B_j(y) - c(y)$. Suppose, to the contrary, that there exists $y' \in \overline{\mathcal{Y}} \backslash \{y^m\}$ such that $y' = \max \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P^*} B_j(y) - c(y) > y^m$. Since $y^m \in \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P^*} B_j(y) - c(y)$, then $\sum_{j \in P^* \cap \mathcal{N}(y^m, y']} \theta_j = c(y') - c(y^m)$. However, by the construction of P^*, $P^* \cap \mathcal{N}(y^m, y'] = \emptyset$, which implies that $0 = \sum_{j \in P^* \cap \mathcal{N}(y^m, y']} \theta_j < c(y') - c(y^m)$. This is a contradiction. $||$

We prove that P^* satisfies *IS* in Claim 2.

Claim 2 For each $i \in P^*$, $y^{P^* \backslash \{i\}} < Y_i$.

Proof of Claim 2. Suppose, to the contrary, that there exists $i \in P^*$ such that $y^{P^* \backslash \{i\}} \geq Y_i$. Let $l \in \{1, \ldots, m\}$ be such that $i \in P^l$. Let $y^k \in \mathcal{Y}$ be such that $y^k = y^{P^* \backslash \{i\}}$. Since $P^l \subseteq \mathcal{N}(y^{l-1}, y^l]$, then $y^{l-1} < Y_i \leq y^l$. Since $y^{l-1} < Y_i$ and $Y_i \leq y^k$, then $l - 1 < k$. Since $\mathcal{Y}$ is discrete and y^l and y^{l-1} lie next to each other, it is impossible that $l > k > l - 1$. Thus, we have $k \geq l > l - 1$. By the construction of P^*, $m > k$.[8]

By the construction of $(\gamma_i)_{i \in N}$ and *GE'*, $y^k = \max \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P^* \backslash \{i\}} B_j(y) - c(y)$. Then, $\sum_{j \in (P^* \backslash \{i\}) \cap \mathcal{N}(y^{l-1}, y^k]} \theta_j \geq c(y^k) - c(y^{l-1})$. Since $P^* \cap \mathcal{N}(y^{l-1}, y^k] = P^l \cup \cdots \cup P^k$,

$$\sum_{j \in (P^* \backslash \{i\}) \cap \mathcal{N}(y^{l-1}, y^k]} \theta_j = \sum_{j \in P^* \cap \mathcal{N}(y^{l-1}, y^k]} \theta_j - \theta_i = \sum_{j \in P^l \cup \cdots \cup P^k} \theta_j - \theta_i.$$

Hence, $\sum_{j \in P^k \cup \cdots \cup P^l} \theta_j - \theta_i \geq c(y^k) - c(y^{l-1})$.

By the construction of P^l, $\sum_{j \in P^m \cup \cdots \cup P^l} \theta_j - \theta_i < c(y^m) - c(y^{l-1})$, which can be rewritten

[8] Suppose, to the contrary, that $y^k = y^m$. By the construction of P^*, $\sum_{j \in P^m \cup \cdots \cup (P^l \backslash \{i\})} \theta_j < c(y^m) - c(y^{l-1})$. Since $y^k = y^m$, then $\sum_{j \in P^k \cup \cdots \cup (P^l \backslash \{i\})} \theta_j < c(y^k) - c(y^{l-1})$. By $y^{P^* \backslash \{i\}} = y^k$, $\sum_{j \in P^k \cup \cdots \cup (P^l \backslash \{i\})} \theta_j \geq c(y^k) - c(y^{l-1})$, which is a contradiction.

as

$$\sum_{j\in P^k\cup\cdots\cup P^l}\theta_j-\theta_i<c(y^m)-c(y^{l-1})-[\sum_{j\in P^m\cup\cdots\cup P^{k+1}}\theta_j]$$

$$=c(y^m)-c(y^k)+c(y^k)-c(y^{l-1})-[\sum_{j\in P^m\cup\cdots\cup P^{k+1}}\theta_j].$$

Since $\sum_{j\in P^m\cup\cdots\cup P^{k+1}}\theta_j\geq c(y^m)-c(y^k)$, then $\sum_{j\in P^k\cup\cdots\cup P^l}\theta_j-\theta_i<c(y^k)-c(y^{l-1})$, which is a contradiction. $||$

For the other levels of the public good $y^k\in\mathcal{Y}$ such that $y^k<y^m$, we can construct a set of participants P such that $y^P=y^k$ and P satisfies *IS* by similar steps. ■

In the proof of Lemma 8, we construct a set of participants that produces y^m units of the public good and from which no participant wants to withdraw. For this construction, we iteratively take an operator "argmin" and construct a set as small as possible. Of course, there are other sets that produce y^m units of the public good. However, if a set of participants is "too large," then there may be a case in which even if one of the members withdraws from the set, her demand level is fulfilled. In such a case, she must want to deviate to not participating. Claim 2 shows that if every participant withdraws from the constructed set, her demand level is not met. That is, the participation of every agent in the set is necessary to fulfill her demand level. Since every member of the constructed set is "pivotal" to fulfill her demand, she has no incentive not to participate.

Since y^m is the maximal possible demand level, P^* in the above proof trivially satisfies *ES*. Therefore, in the induced participation game, under *GE'*, there exists a Nash equilibrium set of participants that provides the public good efficiently. Thus, there is an efficient subgame perfect Nash equilibrium in the voluntary participation game.

Proposition 1 *The participation game has an efficient subgame perfect Nash equilibrium.*

6.2 Existence of inefficient equilibria

The induced participation game may have a Nash equilibrium at which the level of the public good is inefficient. If an internally stable set that produces an inefficient level of the public good satisfies *ES*, then the inefficient level is provided at an equilibrium. While in Example 1, no set of participants that produces the public good inefficiently satisfies *ES*, in Example 2, for each inefficient demand level, there is a set of participants that satisfies *IS* and *ES*.

Example 1 Let $\overline{\mathcal{Y}} = \{\overline{y}^1, \overline{y}^2\}$, with $c(\overline{y}^1) = c(\overline{y}^2) - c(\overline{y}^1) = 10$. Let $N(\overline{y}^1) = \{1\}$ and $N(\overline{y}^2) = \{2\}$, with $\theta_1 = \theta_2 = 12$. Pick up $(\gamma_i)_{i \in N}$ such that the second-stage outcome satisfies *GE*, *IR*, and *PCS* for each set of participants. Then, $\overline{\mathcal{Y}} = \mathcal{Y}$. Set $\{1, 2\}$ is the only the set that satisfies *IS* and *ES*. Thus, No inefficient Nash equilibrium exists.

Example 2 Let $N(\overline{y}^1) = \{1, 2\}$ and $N(\overline{y}^2) = \{3, 4\}$, with $\theta_i = 6$ for each $i \in N$. Suppose that $(\gamma_i)_{i \in N}$ satisfies *GE*, *IR*, and *PCS*. The cost function is the same as that in Example 1. Sets N is a Nash equilibrium set of participants at which $\overline{y}^2$ units of the public good are provided. Set $N(\overline{y}^1)$ and $\{\emptyset\}$ satisfy *IS* and *ES*. Thus, each $y \in \overline{\mathcal{Y}} \cup \{0\}$ is supported at a Nash equilibrium.

Note that $\sum_{j \in N} B_j(y)$ and $c(y)$ are the same across these examples, which implies that the efficient level of the public good is the same in these examples. In this sense, they are equivalent. The difference lies in the benefit of each agent. For each demand level, an agent's benefit in Example 1 is twice as large as in Example 2. We can confirm from these examples that if the benefit of an agent is sufficiently smaller than the marginal cost, then her additional participation is not sufficient to fulfill her demand level. Hence, she is indifferent between participating and not participating. This is the reason why the inefficient Nash equilibrium is likely to exist when the per-capita benefit from the public good is sufficiently small.

Lemma 9 Suppose that *GE'* holds. (i) In the induced participation game, for each $y^l \in \{0\} \cup \mathcal{Y} \backslash \{y^m\}$, if $\theta_i < c(Y_i) - c(y^l)$ for each $i \in N$ such that $Y_i > y^l$, then there is a Nash equilibrium set of participants that produces y^l units of the public good. (ii) If $\theta_i < c(y^{k+1}) - c(y^k)$ for each $y^k \in \mathcal{Y} \cup \{0\}$ and each $i \in \mathcal{N}(y^k, y^{k+1}]$, then for each $y^l \in \mathcal{Y} \cup \{0\}$, there is a Nash equilibrium that produces y^l units of the public good.

Proof. (i) By Lemma 8, $P \subseteq \mathcal{N}(0, y^l]$ can be constructed such that P satisfies *IS* and $y^P = y^l$. When $y^l = 0$, set $P = \emptyset$. For each $i \in N$ such that $Y_i > y^l$, if i additionally joins P, the public good level is unchanged. Hence, P satisfies *ES*.

(ii) Let $y^l \in \mathcal{Y} \cup \{0\}$ and let $i \in N$ be such that $Y_i > y^l$. Denote $y^r \equiv Y_i$. Then, $i \in \mathcal{N}(y^{r-1}, y^r]$ and by the hypothesis, $\theta_i < c(y^r) - c(y^{r-1})$. Since $y^r > y^{r-1} \geq y^l$ and $c(y^{r-1}) \geq c(y^l)$, $\theta_i < c(y^r) - c(y^{r-1}) \leq c(y^r) - c(y^l)$. By (i) of Lemma 9, y^l units of the public good are provided at a Nash equilibrium. ■

Proposition 2 summarizes the results of section 6.2.

Proposition 2 *If the benefit of every agent from the public good is sufficiently small relative to the cost, then the participation game has an inefficient subgame perfect Nash equilibrium. In some cases, for each possible level of the public good, including zero units, there is a subgame perfect Nash equilibrium that produces it.*

Furusawa and Konishi (2011), Healy (2010), and Konishi and Shinohara (2012) show that as the proportion of agents in a population decreases, the level of the public good at an equilibrium diminishes to zero.[9] They study a preference domain on which agents' utilities continuously and monotonically increase with respect to the level of the public good. An intuitive reason for their result is that as a proportion of an agent gets small, she becomes less influential on public good provision: a small agent does not have enough power to change the level of the public good. However, if she participates, then she defrays a cost of a public good. Thus, when the "size" of an agent is sufficiently small, she refrains from participating, which causes the inefficient provision of a public good. This does not apply to our model. Given a

[9] They adopt Milleron's (1972) replication of an economy to investigate this relationship.

set of participants, if there is a non-participant whose demand level is not fulfilled and her switch to participation fulfills her demand, then by *IR*, she obtains a non-negative payoff by the switch. Thus, she is not worse off by participating. The discontinuity of a benefit and *IR* guarantee a participation incentive around the demand level. Note that this applies to the case in which the size of an agent is very small. Thus, at an equilibrium, a positive level of the public good is always provided even if a proportion of an agent becomes infinitely small.

7 Efficient subgame perfect Nash equilibria and coordination

Consider the situation in which agents can coordinate their strategies. What consequences does such coordination lead to? We determine which subgame perfect Nash equilibria can be achieved through coordination, as modeled through a strong perfect equilibrium.

Since the strong perfect equilibrium is stronger than the subgame perfect Nash equilibrium, we first show that the strong perfect equilibrium exists in the voluntary participation game.

Proposition 3 *There is a strong perfect equilibrium in the voluntary participation game.*

The proof is in the appendix.

In the appendix, we construct an efficient subgame perfect Nash equilibrium $s \in \mathcal{S}^n$ at which y^m units of the public good are provided. A feature of s is that

> ($\star$) if a member of $P(s)$ joins in a deviation, her contribution does not decrease whenever her demand level is fulfilled after the deviation.

This feature plays an important role in showing that s is strongly perfect.

We briefly present a reason why no coalition can profitably deviate from s. For this, it is convenient to divide N into $(N \backslash P(s)) \cap \mathcal{N}(0, y^m]$, $P(s) \cap \mathcal{N}(0, y^m]$, and $\mathcal{N}(y^m, \overline{y}^t]$.

At s, every $i \in (N \backslash P(s)) \cap \mathcal{N}(0, y^m]$ chooses not to participate and obtains payoff θ_i at s. Payoff θ_i is the greatest payoff that i can obtain. Thus, if $i \in (N \backslash P(s)) \cap \mathcal{N}(0, y^m]$ joins in the deviation, she is not made better off.

No agent in $\mathcal{N}(y^m, \overline{y}^t]$ joins in the deviation, either. At s, the demand level of every agent in $\mathcal{N}(y^m, \overline{y}^t]$ is not fulfilled and her payoff is zero. If agent $i \in \mathcal{N}(y^m, \overline{y}^t]$ is made better off, her demand level is fulfilled after the deviation: the level of the public good must be greater than or equal to Y_i. It is necessary for the provision of at least Y_i units of the public good that members of the deviation cover at least $c(Y_i) - c(y^m)$. However, since y^m is an efficient level of the public good, the additional benefit from the increase from y^m to Y_i, $\sum_{j \in \mathcal{N}(y^m, Y_i]} \theta_j$, is not greater than $c(Y_i) - c(y^m)$. Hence, if agents in $\mathcal{N}(y^m, Y_i]$ deviate from s, it is impossible to make all of them better off. In order for agents in $\mathcal{N}(y^m, Y_i]$ to be made better off, it is necessary that agents in $\mathcal{N}(y^m, Y_i]$ deviate jointly with at least one agent $j \in P(s) \cap \mathcal{N}(0, y^m]$ and j defrays some portion of $c(Y_i) - c(y^m)$. If such a deviation is done, j's demand level is fulfilled because $Y_i > y^m \geq Y_j$. However, by $(\star)$, j's cost share does not decrease after the deviation. Thus, j is not made better off. It is impossible for a member of $\mathcal{N}(y^m, \overline{y}^t]$ to be made better off.

A deviation among agents in $P(s)$ is also not profitable. If a cost share of some agent decreases, it is necessary to maintain the level of the public good to increase a cost share of another agent in $P(s)$. Thus, deviation by only the internal members of $P(s)$ is not profitable. In conclusion, no coalitional deviation from s can be profitable.

Proposition 4 *Only the efficient subgame perfect Nash equilibrium is a strong perfect equilibrium.*

Proof. Let $s = (s^1, (\gamma_i)_{i \in N}) \in \mathcal{S}$ be an inefficient subgame perfect Nash equilibrium. We show that s is not strongly perfect. Let $\overline{y}^l \in \overline{\mathcal{Y}}$ be the level of the public good that $P(s)$ produces at s. We first consider the case in which $P(s)$ does not provide a public good at its group efficient level.

Case 1. $\overline{y}^l \notin \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P(s)} B_i(y) - c(y)$.

Let $\overline{y}^k = \max\ \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P(s)} B_i(y) - c(y)$. By Lemma 1, $\overline{y}^k > \overline{y}^l$ and $\sum_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]} \theta_i > c(\overline{y}^k) - c(\overline{y}^l)$. By Lemma 1, since s is a Nash equilibrium, $\gamma_i^z(P(s)) = 0$ for each $i \in P(s)$

such that $Y_i > \overline{y}^l$ and each z such that $\overline{y}^z \in \overline{\mathcal{Y}}$. Hence, $\sum_{i \in P(s) \cap \mathcal{N}(0, \overline{y}^l]} \gamma_i^z(P(s)) \geq c(\overline{y}^z) - c(\overline{y}^{z-1})$ for each $z \in \{1, \ldots, l\}$. Since $\sum_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]} \theta_i > c(\overline{y}^k) - c(\overline{y}^l)$, then for each $z \in \{l+1, \ldots, k\}$, there is $(\sigma_i^z)_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]} \in \mathbb{R}_+^{|P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]|}$ such that $\sum_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]} \sigma_i^z = c(\overline{y}^z) - c(\overline{y}^{z-1})$ and $\theta_i - \sum_{r=l+1}^k \sigma_i^r > 0$ for each $i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$. Set $(\gamma'_i(P(s)))_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]}$ such that for each $i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$,

$$\gamma_i'^z(P(s)) = \begin{cases} \sigma_i^z & \text{if } z \in \{l+1, \ldots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$

While the payoff to every agent in $P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$ is zero at $(\gamma_i(P(s)))_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]}$, the payoff to every agent in $P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$ is positive at $(\gamma'_i(P(s)))_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]}$. Thus, if $P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$ deviates from $(\gamma_i(P(s)))_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]}$ to $(\gamma'_i(P(s)))_{i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]}$, then every agent in $P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$ is made better off.

We next consider the case in which $P(s)$ produces a public good at its group efficient level, but the level is not efficient in the economy.

Case 2. $\overline{y}^l \in \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P(s)} B_i(y) - c(y) \setminus \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in N} B_i(y) - c(y)$.

Let $\overline{y}^k = \max \ \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in N} B_i(y) - c(y)$. Then, $\sum_{i \in \mathcal{N}(\overline{y}^l, \overline{y}^k]} \theta_i > c(\overline{y}^k) - c(\overline{y}^l)$. By Lemma 1, $\sum_{z=1}^t \gamma_i^z(P(s)) = 0$ for each $i \in P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$. The payoff to every agent in $P(s) \cap \mathcal{N}(\overline{y}^l, \overline{y}^k]$ is zero.

Denote $D \equiv P(s) \cup \mathcal{N}(\overline{y}^l, \overline{y}^k]$. Set $s'_D = (s'^1, (\gamma'_i)_{i \in D})$ such that $s_i'^1 = 1$ for each $i \in D$. Since $\sum_{i \in \mathcal{N}(\overline{y}^l, \overline{y}^k]} \theta_i > c(\overline{y}^k) - c(\overline{y}^l)$, we can set $(\gamma'_i)_{i \in \mathcal{N}(\overline{y}^l, \overline{y}^k]}$ such that $\sum_{i \in \mathcal{N}(\overline{y}^l, \overline{y}^k]} \gamma_i'^z(D) = c(\overline{y}^z) - c(\overline{y}^{z-1})$ for each $z \in \{l+1, \ldots, k\}$, $\gamma_i'^z(D) = 0$ for each $z > k$, and $\theta_i - \sum_{z=l+1}^k \gamma_i'^z(D) > 0$ for each $i \in \mathcal{N}(\overline{y}^l, \overline{y}^k]$. For each $i \in \mathcal{N}(\overline{y}^l, \overline{y}^k]$, there exists $\sigma_i > 0$ such that $\theta_i - \sum_{z=l+1}^k \gamma_i'^z(D) - \sigma_i > 0$. If every $i \in \mathcal{N}(\overline{y}^l, \overline{y}^k]$ contributes σ_i to the provision of the public good of less than $\overline{y}^l$ units, we can set $(\gamma'_i(D))_{i \in P(s) \cap \mathcal{N}(0, \overline{y}^l]}$ such that $\gamma_i'^z(D) \leq \gamma_i^z(P(s))$ for each $i \in P(s) \cap \mathcal{N}(0, \overline{y}^l]$ and each $z \in \{1, \ldots, l\}$ with strict inequality for at least one z, $\sum_{i \in D} \gamma_i'^z(D) = c(\overline{y}^z) - c(\overline{y}^{z-1})$ for each $z \in \{1, \ldots, l\}$, and $\gamma_i'^z(D) = 0$ for each $z > l$. If D deviates from s_D to s'_D, then every agent in D is made better off.

In Cases 1 and 2, s is unstable against a coalitional deviation. ■

8 Extention: Multiple thresholds

In the real world, there may be a situation in which an individual has multiple thresholds and her benefit from a public good gradually increases as the threshold is fulfilled. We extend the analysis to such a case and examine whether the participation game has a subgame perfect Nash equilibrium to produce an efficient allocation.

We introduce a benefit function of each agent in such a way that her benefit can have more than one threshold. Let $\overline{\mathcal{Y}} = \{\overline{y}^1, \ldots, \overline{y}^t\}$, where t is a positive integer, be the set of thresholds at which the benefit of an agent in the economy increases. Let $\theta_i^k \geq 0$ be an additional benefit of agent $i \in N$ at the level $\overline{y}^k \in \overline{\mathcal{Y}}$. To define a benefit function consistently, we define $\theta_i^0 \equiv 0$ for each $i \in N$. The benefit function of $i \in N$, $B_i(y)$, is defined as follows: $B_i(y) = \sum_{k=0}^{l-1} \theta_i^k$ if $y \in \left[\overline{y}^{l-1}, \overline{y}^l\right)$, where $1 \leq l \leq t$. The payoff function of $i \in N$ is similar: $V_i(y, x_i) = B_i(y) - x_i$. The cost function is the same as in the previous sections. We can define the voluntary participation game similarly.[10] Note that if for each $i \in N$ there is the unique $k \in \{1, \ldots, t\}$ such that $\theta_i^k > 0$, then the model is the same as in the previous sections.

We first provide an example in which agents have multiple thresholds and no equilibrium supports the efficient provision of a public good. Thus, if agents have a demand level but their thresholds are more than one, then a public good may be under-provided in the voluntary participation game.

Example 3 Let $N = \{1, 2\}$. Let $\overline{\mathcal{Y}} = \{\overline{y}^1, \overline{y}^2\}$ be such that $\overline{y}^1 < \overline{y}^2$. Let $\left(\theta_1^1, \theta_1^2\right) = (12, 9)$ and $\left(\theta_2^1, \theta_2^2\right) = (12, 2)$. Both agents have the same thresholds. Let $c(\overline{y}^1) = 10$ and $c(\overline{y}^2) = 20$. In this case, $\overline{y}^2$ is the only efficient level of the public good. We show that there is no subgame perfect Nash equilibrium on the path of which $\overline{y}^2$ units of the public good are provided. We start with the analysis of the second stage. When only one agent participates, $\overline{y}^1$ units of the public good are provided at a Nash equilibrium. When both agents participate, $\overline{y}^1$ and $\overline{y}^2$

[10] We assume that the second stage of the voluntary participation game is the same as that in the previous analysis. Note that the second-stage contribution game can be applied to this extended model.

are supported at a Nash equilibrium in the contribution game.[11] Since N is the only set of participants that produces $\overline{y}^2$ at an equilibrium, if N provides $\overline{y}^1$ units of the public good, then $\overline{y}^2$ units of the public good can not be provided. Hence, to examine whether $\overline{y}^2$ units of the public good are provided at an equilibrium, we necessarily assume that N provides $\overline{y}^2$ units of the public good at a Nash equilibrium. Since N provides $\overline{y}^2$, then $\sum_{i\in N}(\gamma_i^1(N) + \gamma_i^2(N)) \geq c(\overline{y}^2)$. Given this second-stage equilibrium, if both agents participate at a Nash equilibrium, then $B_i(\overline{y}^2) - \big(\gamma_i^1(N) + \gamma_i^2(N)\big) \geq B_i(\overline{y}^1)$ for each $i \in N$. Thus, $\theta_{i2} \geq \gamma_i^1(N) + \gamma_i^2(N)$ for each $i \in N$. Summing up this condition over $i \in N$ yields $\sum_{i\in N}\theta_{i2} \geq \sum_{i\in N}(\gamma_i^1(N)+\gamma_i^2(N))$. Since $\sum_{i\in N}(\gamma_i^1(N) + \gamma_i^2(N)) \geq c(\overline{y}^2)$, we have $\sum_{i\in N}\theta_{i2} = 11 \geq 20$, which is impossible. Therefore, no subgame perfect Nash equilibrium exists such that the public good is provided efficiently. Of course, no strong perfect equilibrium exists.

We next provide an example in which every agent has two thresholds that are not the same across the agents. In this example, there is a subgame perfect Nash equilibrium that produces an efficient allocation.

Example 4 Let $N = \{1, 2\}$. Let $\overline{\mathcal{Y}} = \{\overline{y}^1, \overline{y}^2, \overline{y}^3\}$ be such that $\overline{y}^1 < \overline{y}^2 < \overline{y}^3$. Let $\big(\theta_1^1, \theta_1^2, \theta_1^3\big) = (12, 6, 0)$ and $\big(\theta_2^1, \theta_2^2, \theta_2^3\big) = (0, 6, 12)$. Let $c(\overline{y}^1) = 10$, $c(\overline{y}^2) = 20$, and $c(\overline{y}^3) = 30$. In this case, $\overline{y}^3$ is the only efficient level of the public good. We can check that the following strategy profile $(s^1, (\gamma_i)_{i\in N})$ is a subgame perfect Nash equilibrium on the path of which $\overline{y}^3$ units of the public good are provided:

- $s_1^1 = s_2^1 = 1$ (both agents participate)
- For each $i \in N$, $\big(\gamma_i^k(P)\big)_{k\in\{1,2,3\}}$ such that $P \subseteq N$ satisfies

$$\big(\gamma_1^k(P)\big)_{k\in\{1,2,3\}} = \begin{cases}(10,0,0) & \text{if } P=\{1\}\\ (10,5,0) & \text{if } P=N\end{cases} \text{ and } \big(\gamma_2^k(P)\big)_{k\in\{1,2,3\}} = \begin{cases}(0,0,0) & \text{if } P=\{2\}\\ (0,5,10) & \text{if } P=N\end{cases}.$$

[11] For example, $\big(\gamma_1^1(N), \gamma_1^2(N)\big) = (5, 8.5)$ and $\big(\gamma_2^1(N), \gamma_2^2(N)\big) = (5, 1.5)$ constitute a Nash equilibrium at which $\overline{y}^2$ units of the public good are provided, and $\big(\gamma_1^1(N), \gamma_1^2(N)\big) = (0, 0)$ and $\big(\gamma_2^1(N), \gamma_2^2(N)\big) = (10, 0)$ constitute a Nash equilibrium at which $\overline{y}^1$ units of the public good are produced.

We explain the reason why there is not necessarily a subgame perfect equilibrium set of participants that produces a public good efficiently in the model of multiple thresholds. In order to explain it, we must consider what the benefits and the costs of participating are. Consider a situation in which $i \in N$ decides whether she joins in $P\backslash\{i\} \subsetneq N$. If i does not join in $P\backslash\{i\}$, then i can free-ride the public good provided by $P\backslash\{i\}$, denoted by $y^{P\backslash\{i\}}$. If i joins, then the level of a public good, denoted by y^P, may be greater than $y^{P\backslash\{i\}}$. If $y^P > y^{P\backslash\{i\}}$, i's (gross) benefit from y^P is greater than the benefit from $y^{P\backslash\{i\}}$ in the multiple-threshold model. The additional benefit from participating is the difference in benefits. If $i \in N$ joins in $P\backslash\{i\}$, she shares the cost of the public good. Let x_i^P be this cost burden. This is the cost of participating. Therefore, $i \in N$ joins in $P\backslash\{i\}$ only if

$$B_i(y^P) - B_i(y^{P\backslash\{i\}}) \geq x_i^P. \tag{6}$$

Summing up (6) over $i \in P$ yields

$$\sum_{i \in P} \left[B_i(y^P) - B_i(y^{P\backslash\{i\}})\right] \geq \sum_{i \in P} x_i^P \geq c(y^P). \tag{7}$$

The last inequality follows from the budget feasibility. Therefore, (7) is a necessary condition for $P \subseteq N$ to be supported at an equilibrium. This condition requires that the sum of *additional* benefits of the participants covers the cost of the public good.

In the case in which each agent has the only threshold, there is a set of participants that satisfies (7) and provides a public good efficiently. Set P^*, which is constructed in the proof of Lemma 8, is such an example. For each $i \in P^*$, $y^{P^*\backslash\{i\}} < Y_i$; hence, (7) is equivalent to $\sum_{i \in P^*} B_i(y^{P^*}) \geq c(y^{P^*})$. Since P^* provides a group efficient level of a public good for P^*, $\sum_{i \in P^*} B_i(y^{P^*}) \geq c(y^{P^*})$. Thus, *GE* for P^* implies that P^* satisfies (7). In the case in which each agent has multiple thresholds, *GE* also implies that $\sum_{i \in P} B_i(y^P) \geq c(y^P)$. However, since $B_i(y^{P\backslash\{i\}})$ may not be zero for some $P \subseteq N$ and some $i \in P$, it does not generally imply (7).[12] Thus, in this case, even if each $P \subseteq N$ provides a group efficient level of a public good, there may not be an equilibrium set of participants that provides the public good efficiently.

[12] While $B_i(y^{P\backslash\{i\}}) > 0$ for each $i \in N$ in Example 3, it is equal to zero for each $i \in N$ in Example 4.

It is also worth noting that (6) is equivalent to *IR* for P^* when each agent has a single threshold: *IR* for P^* guarantees the participation incentive. In the case in which each agent has multiple thresholds, if a set of participants is supported at a Nash equilibrium, then an incremental benefit from participating (not a *whole* benefit) covers the cost burden of each participant. This condition is stronger than *IR*. As in Example 3, if the participation of an agent whose incremental benefit is small is necessary to provide a public good efficiently, then the public good is unlikely to be produced efficiently because the small incremental benefit may not cover the payment, which is distributed based on the total, not incremental, cost of the public good.

9 Concluding remarks

We examine the participation problem in public good provision when agents have demand levels for the public good. In our model, agents can benefit from a public good if the public good is provided at a level greater than their demand level. First, we show that there exists a subgame perfect Nash equilibrium on the path of which the public good is produced efficiently. Second, we confirm that there may exist a subgame perfect Nash equilibrium at which the public good is produced inefficiently. Third, coordination modeled through a strong perfect equilibrium singles out the efficient subgame perfect Nash equilibrium. Fourth, if agents have multiple thresholds, then there may be no subgame perfect Nash equilibrium at which the public good is produced efficiently.

Earlier studies have pointed out that the voluntary participation problem is serious. However, when each agent has a single demand level, the efficient provision of the public good is supported at a subgame perfect Nash equilibrium and only the efficient provision of the public good is attained through coordination. Hence, we can conclude that the problem is not as serious as the earlier studies report in cases in which each agent has a single threshold. However, in other cases, the participation problem may be serious. In such cases, we must construct a mechanism that induces voluntary participation and the efficient provision of public goods to solve the problem, such as the unit-by-unit participation mechanism of Nishimura and Shinohara (2011). The construction of a mechanism that can be applied to our model is left for future work.

Appendix: Proof of Proposition 3

We first construct a strategy profile and then show that it is a strong perfect equilibrium.

Suppose that $y^m \in \mathcal{Y}$ corresponds to $\overline{y}^m \in \overline{\mathcal{Y}}$. Let $P^* \subseteq N$ be a set of participants that produces $\overline{y}^m$ units of the public good and is constructed according to the method of the proof of Lemma 8. For each $i \in P^*$, let $\boldsymbol{g_i} = (g_i^z)_{z=1}^t \in \mathbb{R}_+^t$ be a vector of marginal contributions that is constructed in Section 4: (i) for each $\overline{y}^z \in \overline{\mathcal{Y}}$ such that $\overline{y}^z > Y_i$, $g_i^z = 0$, (ii) for each $z \in \{1, \dots, m\}$ and each $i \in P^* \cap \mathcal{N}(\overline{y}^{z-1}, \overline{y}^m]$, $\sum_{i \in P^* \cap \mathcal{N}(\overline{y}^{z-1}, \overline{y}^m]} g_i^z = c(\overline{y}^z) - c(\overline{y}^{z-1})$ and $\theta_i - \sum_{k=z}^m g_i^k \geq 0$, and (iii) for each $i \in P^*$, $\sum_{z=1}^m g_i^z > 0$. We can take such $\boldsymbol{g_i}$ in a manner similar to the proof of Lemma 2.

Let $s = (s^1, (\gamma_i)_{i \in N}) \in \mathcal{S}$ be such that $P(s) = P^*$, constructed in Lemma 8, and $\gamma_i(P^*) = \boldsymbol{g_i}$ for each $i \in P^*$. Define $(\gamma_i(P))_{i \in P}$ for each $P \subseteq N$ such that $P \neq P^*$ as follows:

(a) If $P \cap P^* = \emptyset$, then $(\gamma_i(P))_{i \in P}$ is an arbitrary profile of a vector of marginal contributions that is supported at a strong Nash equilibrium of the contribution game when P is a set of participants.

(b) If $P^* \subseteq P$, then for each $i \in P$,

$$\gamma_i(P) \equiv \begin{cases} \boldsymbol{g_i} & \text{if } i \in P^* \\ (0, \dots, 0) \in \mathbb{R}_+^t & \text{otherwise} \end{cases}.$$

(c) Let $\overline{y}^l = \max \ \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P} B_i(y) - c(y)$. If $P \cap P^* \neq \emptyset$, but not $P^* \subseteq P$, then $(\gamma_i(P))_{i \in P}$ is defined as follows:

(c.1) For each $i \in P$ such that $Y_i > \overline{y}^l$ and each $z \in \{1, \dots, t\}$, $\gamma_i^z(P) \equiv 0$. For each $i \in P \cap \mathcal{N}(0, \overline{y}^l]$ and each $z \in \{1, \dots, t\}$, if $\overline{y}^z > Y_i$, then $\gamma_i^z(P) \equiv 0$.

(c.2) For each $k \in \{1, \dots, l\}$ and each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]$, $\theta_i - \sum_{z=k}^t \gamma_i^z(P) \geq 0$ and $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]} \gamma_i^k(P) = c(\overline{y}^k) - c(\overline{y}^{k-1})$.

(c.3) For each $k \in \{1, \ldots, l\}$ and each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*$, $\sum_{z=k}^{l} \gamma_i^z(P) \geq \sum_{z=k}^{l} g_i^z$.[13]

Clearly, a vector of marginal contributions that satisfies (a) and (b) can be constructed. Properties (c.1) and (c.2) are the same as (a) and (b) in Section 4. In this section, in addition to these conditions, we require (c.3). This property means that if a set of participants switches from P^* to P such that $P \cap P^* \neq \emptyset$ but not $P^* \subseteq P$, then the common members between these sets whose demand level is met contribute at least the level before the switch. This property plays an important role in proving that s is a strong perfect equilibrium. We first show that a vector of marginal contributions can be set in a way that satisfies (c) for each $P \subseteq N$ such that $P \cap P^* \neq \emptyset$ but not $P^* \subseteq P$.

Lemma 10 For each $P \subseteq N$ such that $P \cap P^* \neq \emptyset$, but not $P^* \subseteq P$, there is a vector of marginal contributions that satisfies (c).

Proof. We show by induction. Obviously, $(\gamma_i)_{i \in P}$ can be constructed in a way that satisfies (c.1). We first consider the case of $k = l$. If $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \setminus P^*} \theta_i \geq c(\overline{y}^l) - c(\overline{y}^{l-1}) - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*} g_i^l$, then $\gamma_i^l(P) = g_i^l$ for each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*$ and define $(\gamma_i^l(P))_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \setminus P^*}$ such that $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \setminus P^*} \gamma_i^l(P) = c(\overline{y}^l) - c(\overline{y}^{l-1}) - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*} g_i^l$ and $\theta_i - \gamma_i^l(P) \geq 0$ for each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \setminus P^*$. Otherwise, by *GE*, $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l]} \theta_i \geq c(\overline{y}^l) - c(\overline{y}^{l-1})$. By this condition,

$$\sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*} (\theta_i - g_i^l) \geq c(\overline{y}^l) - c(\overline{y}^{l-1}) - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*} g_i^l - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \setminus P^*} \theta_i > 0.$$

For each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*$, define $\varepsilon_i^l \geq 0$ as

$$\sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*} (\theta_i - g_i^l) \geq \sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*} \varepsilon_i^l = c(\overline{y}^l) - c(\overline{y}^{l-1}) - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*} g_i^l - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \setminus P^*} \theta_i$$

[13] Note that $\sum_{z=1}^{t} \gamma_i^z(P) = \sum_{z=1}^{l} \gamma_i^z(P) \geq \sum_{z=1}^{l} \gamma_i^z(P^*) = \sum_{z=1}^{t} \gamma_i^z(P^*)$ for each $i \in P \cap \mathcal{N}(0, \overline{y}^l] \cap P^*$.

and $\theta_i - g_i^l \geq \varepsilon_i^l$ for each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*$. Set $\gamma_i^l(P) = g_i^l + \varepsilon_i^l$ for each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*$ and $\gamma_i^l(P) = \theta_i$ for each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \backslash P^*$. In conclusion, in any case, $\theta_i - \gamma_i^l(P) \geq 0$ for each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l]$, $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l]} \gamma_i^l(P) = c(\overline{y}^l) - c(\overline{y}^{l-1})$, and $\gamma_i^l(P) \geq g_i^l$ for each $i \in P \cap \mathcal{N}(\overline{y}^{l-1}, \overline{y}^l] \cap P^*$.

Let $k \in \{1, \ldots, l-1\}$. Suppose that $\gamma_i^z(P)$ is defined in a way that satisfies (c.1) – (c.3) for each $i \in P$ and each $z \in (k, l]$. We now construct $\gamma_i^k(P)$ for each $i \in P$. Define $\gamma_i^k(P) = 0$ for each $i \in P \cap \mathcal{N}(0, \overline{y}^{k-1}]$. If we set $\gamma_i^k(P)$ for each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*$ in such a way that $\gamma_i^k(P) \geq \max\{0, g_i^k - \sum_{z=k+1}^l (\gamma_i^z(P) - g_i^z)\}$, then $\sum_{z=k}^l \gamma_i^z(P) \geq \sum_{z=k}^l g_i^z$, which is shown in Claim 3. Hence, (c.3) holds for k.

Claim 3 For each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*$, if $\gamma_i^k(P) \geq \max\{0, g_i^k - \sum_{z=k+1}^l (\gamma_i^z(P) - g_i^z)\}$, then $\sum_{z=k}^l \gamma_i^z(P) \geq \sum_{z=k}^l g_i^z$.

Proof of Claim 3. If $g_i^k - \sum_{z=k+1}^l (\gamma_i^z(P) - g_i^z) \geq 0$, then it is trivial. If $g_i^k - \sum_{z=k+1}^l (\gamma_i^z(P) - g_i^z) < 0$, then $\sum_{z=k+1}^l \gamma_i^z(P) > \sum_{z=k}^l g_i^z$. Since $\gamma_i^k(P) \geq 0$, then $\sum_{z=k}^l \gamma_i^z(P) \geq \sum_{z=k+1}^l \gamma_i^z(P) > \sum_{z=k}^l g_i^z$. ||

Denote $\Delta \equiv c(\overline{y}^k) - c(\overline{y}^{k-1}) - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*} \max\{0, g_i^k - \sum_{z=k+1}^l (\gamma_i^k(P) - g_i^z)\}$.

Case 1. $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \backslash P^*} (\theta_i - \sum_{z=k+1}^l \gamma_i^z(P)) \geq \Delta$.

Denote $\mathcal{X} \equiv \{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^* |\, g_i^k - \sum_{z=k+1}^l (\gamma_i^z(P) - g_i^z) > 0\}$. Then,

$$\Delta = \underbrace{c(\overline{y}^k) - c(\overline{y}^{k-1}) - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^* \cap \mathcal{X}} g_i^k}_{(\alpha)} + \underbrace{\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^* \cap \mathcal{X}} \sum_{z=k+1}^{l} (\gamma_i^z(P) - g_i^z)}_{(\beta)}.$$

By the definitions of $\boldsymbol{g_i}$ and the condition that $P^* \subsetneq P$ does not satisfy, we have

$$c(\overline{y}^k) - c(\overline{y}^{k-1}) = \sum_{i \in \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*} g_i^k > \sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*} g_i^k \geq \sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^* \cap \mathcal{X}} g_i^k.$$

Thus, $(\alpha) > 0$. By the induction hypothesis, $(\beta) \geq 0$. Hence, $\Delta > 0$.

Define $\gamma_i^k(P)$ for each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]\backslash P^*$ such that $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]\backslash P^*} \gamma_i^k(P) = \Delta$ and $\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P) \geq \gamma_i^k(P)$ for each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]\backslash P^*$. Set $\gamma_i^k(P) = \max\{0, g_i^k - \sum_{z=k+1}^{l}(\gamma_i^z(P) - g_i^z)\}$ for each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*$. For each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*$, if $g_i^k - \sum_{z=k+1}^{l}(\gamma_i^z(P) - g_i^z) > 0$, then $\theta_i - \sum_{z=k}^{l} \gamma_i^z(P) = \theta_i - \sum_{z=k}^{l} g_i^z \geq 0$. Otherwise, $\theta_i - \sum_{z=k}^{l} \gamma_i^z(P) = \theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P) \geq 0$ by the induction hypothesis.

Case 2. $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]\backslash P^*} (\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P)) < \Delta$.

By *GE*, $\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]} \theta_i \geq c(\overline{y}^l) - c(\overline{y}^{k-1})$. By this condition,

$$\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]} \theta_i - (c(\overline{y}^l) - c(\overline{y}^k)) \geq c(\overline{y}^k) - c(\overline{y}^{k-1}).$$

By the induction hypothesis,

$$\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]} \left[\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P)\right] \geq c(\overline{y}^k) - c(\overline{y}^{k-1}),$$

which is equal to

$$\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*} \left[\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P)\right] \geq c(\overline{y}^k) - c(\overline{y}^{k-1}) - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]\backslash P^*} \left[\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P)\right].$$

We finally have

$$\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*} \left[\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P) - \max\left\{0, g_i^k - \sum_{z=k+1}^{l} (\gamma_i^k(P) - g_i^z)\right\}\right]$$
$$\geq \Delta - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l]\backslash P^*} \left[\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P)\right].$$

For each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*$, define $\varepsilon_i^k \geq 0$ as

$$\begin{aligned}
&\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*} \left[\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P) - \max\left\{0, g_i^k - \sum_{z=k+1}^{l} (\gamma_i^k(P) - g_i^z)\right\}\right] \\
\geq &\sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*} \varepsilon_i^k \\
&= \Delta - \sum_{i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \backslash P^*} \left[\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P)\right]
\end{aligned}$$

and $\theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P) - \max\left\{0, g_i^k - \sum_{z=k+1}^{l} (\gamma_i^k(P) - g_i^z)\right\} \geq \varepsilon_i^k$ for each $i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^*$. Let

$$\gamma_i^k(P) = \begin{cases} \theta_i - \sum_{z=k+1}^{l} \gamma_i^z(P) & \text{for each } i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \backslash P^* \\ \max\left\{0, g_i^k - \sum_{z=k+1}^{l} (\gamma_i^k(P) - g_i^z)\right\} + \varepsilon_i^k & \text{for each } i \in P \cap \mathcal{N}(\overline{y}^{k-1}, \overline{y}^l] \cap P^* \end{cases}.$$

In any case, the vector constructed satisfies (c.1) – (c.3). ■

Note that by Lemma 4, $(\gamma_i)_{i \in N}$ assigns a strong Nash equilibrium for each set of participants. Note also that s is one of the efficient subgame perfect Nash equilibria. We now show that s is a strong perfect equilibrium; thus, there is a strong perfect equilibrium in the voluntary participation game.

Lemma 11 Strategy s is a strong perfect equilibrium.

Proof. At s,

$$U_i(s) = \begin{cases} \theta_i - \sum_{z=1}^{m} \gamma_i^z(P(s)) \geq 0 \text{ for each } i \in P(s) \cap \mathcal{N}(0, \overline{y}^m] \\ \theta_i \text{ for each } i \in (N \backslash P(s)) \cap \mathcal{N}(0, \overline{y}^m] \\ 0 \text{ for each } i \in \mathcal{N}(\overline{y}^m, \overline{y}^t] \end{cases}.$$

Suppose, to the contrary, that s is not a strong perfect equilibrium. Then, there are $D \subseteq N$ and $s'_D = (s'^1_D, (\gamma'_i)_{i \in D}) \in \mathcal{S}_D$ such that $U_i(s'_D, s_{-D}) > U_i(s)$ for each $i \in D$. Let us define $s' \equiv (s'_D, s_{-D})$. Note that $N = \mathcal{N}(\overline{y}^m, \overline{y}^t] \cup (\mathcal{N}(0, \overline{y}^m] \cap P(s)) \cup (\mathcal{N}(0, \overline{y}^m] \backslash P(s))$.

Claim 4 $D \cap \mathcal{N}(0, \overline{y}^m] \backslash P(s) = \emptyset$.

Proof of Claim 4. At s, each $i \in \mathcal{N}(0, \overline{y}^m] \backslash P(s)$ receives the payoff θ_i, which is the greatest payoff that i can obtain. Thus, even if i joins in the deviation of D, then she is not made better off. ||

Claim 5 $D \cap \mathcal{N}(\overline{y}^m, \overline{y}^t] = \emptyset$.

Proof of Claim 5. Suppose, to the contrary, that $D \cap \mathcal{N}(\overline{y}^m, \overline{y}^t] \neq \emptyset$. Let $\overline{y}^k = \max_{i \in D \cap \mathcal{N}(\overline{y}^m, \overline{y}^t]} Y_i$. Note that if each $i \in D$ such that $Y_i = \overline{y}^k$ is made better off, then $P(s')$ provides at least $\overline{y}^k$ units of the public good. We assume that $P(s')$ provides $\overline{y}^k$ units of the public good at s.[14] If $P(s')$ provides $\overline{y}^k$ units of the public good, it contributes $c(\overline{y}^k)$ in total. Since $\overline{y}^m$ is an efficient level of the public good and it is the maximal efficient demand level,

$$\sum_{i \in \mathcal{N}(\overline{y}^m, \overline{y}^k] \cap D} \theta_i \leq \sum_{i \in \mathcal{N}(\overline{y}^m, \overline{y}^k]} \theta_i < c(\overline{y}^k) - c(\overline{y}^m). \tag{8}$$

If $D \subseteq \mathcal{N}(\overline{y}^m, \overline{y}^k]$, then $P(s) \subseteq P(s')$. By (8), $\overline{y}^m = \max \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{j \in P(s')} B_j(y) - c(y)$. By *GE'*, $P(s')$ produces $\overline{y}^m$ units of the public good at s. By the construction of $(\gamma_i)_{i \in P(s')}$, $\gamma_i^z(P(s')) = 0$ for each $i \in P(s') \cap \mathcal{N}(0, \overline{y}^m]$ and each z such that $\overline{y}^z > \overline{y}^m$.[15] Then, if $\overline{y}^k$ units of the public good are produced, then members of D cover $c(\overline{y}^k) - c(\overline{y}^m)$, which is greater than $\sum_{i \in \mathcal{N}(\overline{y}^m, \overline{y}^k] \cap D} \theta_i$ by (8). Thus, if $D \subseteq \mathcal{N}(\overline{y}^m, \overline{y}^k]$, it is impossible that every $i \in D$ is made better off.

By Claim 4, if there is $i \in D$ such that $i \notin \mathcal{N}(\overline{y}^m, \overline{y}^k]$, then $i \in D \cap P(s) \cap \mathcal{N}(0, \overline{y}^m]$. There must be $i \in D \cap P(s) \cap \mathcal{N}(0, \overline{y}^m]$ such that $s_i'^1 = 1$; otherwise, for each $i \in D \cap P(s) \cap \mathcal{N}(0, \overline{y}^m]$, $s_i'^1 = 0$, which implies that $D \cap P(s') \cap \mathcal{N}(0, \overline{y}^m] = \emptyset$. In this case, agents in $D \cap P(s') \cap \mathcal{N}(\overline{y}^m, \overline{y}^k]$ must pay a portion of $c(\overline{y}^m)$ in addition to $c(\overline{y}^k) - c(\overline{y}^m)$ to produce $\overline{y}^k$ units of the public good. Then, by (8), it is impossible that every $i \in D \cap \mathcal{N}(\overline{y}^m, \overline{y}^k]$ is made better off. Thus, $P(s) \cap P(s') \cap \mathcal{N}(0, \overline{y}^m] \cap D \neq \emptyset$.

Let $\overline{y}^l = \max\ \arg\max_{y \in \overline{\mathcal{Y}}} \sum_{i \in P(s')} B_i(y) - c(y)$. Note that $P(s')$ produces $\overline{y}^l$ at s and $\overline{y}^k$ at s'. Since $\overline{y}^m$ is the maximal efficient demand level, then $\overline{y}^l \leq \overline{y}^m$. Thus, $\overline{y}^l \leq \overline{y}^m < \overline{y}^k$.

[14] We can apply similar logic when $P(s')$ produces a public good at a level higher than $\overline{y}^k$.
[15] Note that $P(s) \cap \mathcal{N}(0, \overline{y}^m] = P(s') \cap \mathcal{N}(0, \overline{y}^m]$.

Since $s_{-D} = s'_{-D}$, $D \cap P(s')$ must contribute $c(\overline{y}^l) - \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \setminus D} \sum_{z=1}^{l} \gamma_i^z(P(s'))$ to the provision of $\overline{y}^l$ units of the public good and $c(\overline{y}^k) - c(\overline{y}^l)$ to increase a public good from $\overline{y}^l$ to $\overline{y}^k$ units. Hence,

$$\sum_{i \in P(s') \cap D} \sum_{z=1}^{t} \gamma_i'^z(P(s')) \geq c(\overline{y}^l) - \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \setminus D} \sum_{z=1}^{l} \gamma_i^z(P(s')) + c(\overline{y}^k) - c(\overline{y}^l). \tag{9}$$

By the construction of $(\gamma_i)_{i \in P(s')}$,

$$\begin{aligned} \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D} \sum_{z=1}^{t} \gamma_i^z(P(s')) &= \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D} \sum_{z=1}^{l} \gamma_i^z(P(s')) \\ &= c(\overline{y}^l) - \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \setminus D} \sum_{z=1}^{l} \gamma_i^z(P(s')). \end{aligned} \tag{10}$$

By Claim 4, $P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D = P(s') \cap P(s) \cap \mathcal{N}(0,\overline{y}^l] \cap D$.[16] Hence, by the construction of $(\gamma_i)_{i \in P(s')}$,

$$\sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D} \sum_{z=1}^{t} \gamma_i^z(P(s')) \geq \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D} \sum_{z=1}^{t} \gamma_i^z(P(s)). \tag{11}$$

For the deviation to be profitable,

$$\sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D} \sum_{z=1}^{t} \gamma_i^z(P(s)) > \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D} \sum_{z=1}^{t} \gamma_i'^z(P(s')). \tag{12}$$

By the property of $\overline{y}^l$,

$$\sum_{i \in P(s') \cap \mathcal{N}(\overline{y}^l,\overline{y}^k] \cap D} \theta_i \leq \sum_{i \in P(s') \cap \mathcal{N}(\overline{y}^l,\overline{y}^k]} \theta_i < c(\overline{y}^k) - c(\overline{y}^l). \tag{13}$$

By (9) and (10),

$$\sum_{i \in P(s') \cap D} \sum_{z=1}^{t} \gamma_i'^z(P(s')) \geq \sum_{i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D} \sum_{z=1}^{t} \gamma_i^z(P(s')) + c(\overline{y}^k) - c(\overline{y}^l).$$

[16] It is trivial that $P(s') \cap P(s) \cap \mathcal{N}(0,\overline{y}^l] \cap D \subseteq P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D$. Let $i \in P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D$. Since $\mathcal{N}(0,\overline{y}^l] \subseteq \mathcal{N}(0,\overline{y}^m]$, $i \in D \cap \mathcal{N}(0,\overline{y}^m]$. By Claim 4, $i \in P(s)$. Thus, $P(s') \cap \mathcal{N}(0,\overline{y}^l] \cap D \subseteq P(s') \cap P(s) \cap \mathcal{N}(0,\overline{y}^l]$.

By (11) and (13),

$$\sum_{i\in P(s')\cap\mathcal{N}(0,\overline{y}^l]\cap D}\sum_{z=1}^{t}\gamma_i^z(P(s'))+c(\overline{y}^k)-c(\overline{y}^l)$$

$$>\sum_{i\in P(s')\cap\mathcal{N}(0,\overline{y}^l]\cap D}\sum_{z=1}^{t}\gamma_i^z(P(s))+\sum_{i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D}\theta_i.$$

By (12),

$$\sum_{i\in P(s')\cap\mathcal{N}(0,\overline{y}^l]\cap D}\sum_{z=1}^{t}\gamma_i^z(P(s))+\sum_{i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D}\theta_i$$

$$>\sum_{i\in P(s')\cap\mathcal{N}(0,\overline{y}^l]\cap D}\sum_{z=1}^{t}\gamma_i'^z(P(s'))+\sum_{i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D}\theta_i.$$

In conclusion,

$$\sum_{i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D}\sum_{z=1}^{t}\gamma_i'^z(P(s'))>\sum_{i\in P(s')\cap\mathcal{N}(0,\overline{y}^l]\cap D}\sum_{z=1}^{t}\gamma_i'^z(P(s'))+\sum_{i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D}\theta_i,$$

which implies that

$$\sum_{i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D}\sum_{z=1}^{t}\gamma_i'^z(P(s'))>\sum_{i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D}\theta_i. \quad (14)$$

By (14), there is $i\in P(s')\cap\mathcal{N}(\overline{y}^l,\overline{y}^k]\cap D$ such that $\theta_i-\sum_{z=1}^{t}\gamma_i'^z(P(s'))<0$. Since $U_i(s)=0$ for such an i, i is made worse off by the deviation, which is a contradiction.

Therefore, $D\cap\mathcal{N}(\overline{y}^m,\overline{y}^t]=\emptyset$. ||

Claim 6 $D\cap\mathcal{N}(0,\overline{y}^m]\cap P(s)=\emptyset$.

Proof of Claim 6. Suppose that $D\cap\mathcal{N}(0,\overline{y}^m]\cap P(s)\neq\emptyset$. By Claims 4 and 5, $D\subseteq P(s)\cap\mathcal{N}(0,\overline{y}^m]$. Then, $P(s')\subseteq P(s)$. Note that if the deviation of D improves the payoff to all members of D, then all of their demand levels are fulfilled. First, consider the case of $P(s')=P(s)$: that is, $s_i'^1=1$ for each $i\in D$. By the construction of $(\gamma_i)_{i\in N}$, if $j\in D$ reduces her contribution, then j's demand level is not fulfilled. Then, agent $k\in D\backslash\{j\}$ must increase her contribution to fulfill j's demand level, which implies that k is made worse off.

Second, consider the case of $P(s') \subsetneq P(s)$: that is, $s_i'^1 = 0$ for some $i \in D$. Since $P(s)$ satisfies *IS*, then $\overline{y}^q \leq \overline{y}^r < Y_i$, in which $\overline{y}^q \equiv \max\ \arg\max_{y\in\overline{\mathcal{Y}}} \sum_{j\in P(s')} B_j(y) - c(y)$ and $\overline{y}^r \equiv \max\ \arg\max_{y\in\overline{\mathcal{Y}}} \sum_{j\in P(s)\setminus\{i\}} B_j(y) - c(y)$. Since $P(s')$ produces $\overline{y}^q$ units of the public good and $s_{N\setminus D} = s'_{N\setminus D}$, then $D \cap P(s')$ must pay $\sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{q} \gamma_j^z(P(s'))$ to produce $\overline{y}^q$ units of the public good at s'. It also pays $c(Y_i) - c(\overline{y}^q)$ to increase the public good from $\overline{y}^q$ to Y_i. If $P(s')$ produces Y_i units of the public good at s', then $D \cap P(s')$ pays at least $c(Y_i) - c(\overline{y}^q) + \sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{q} \gamma_j^z(P(s'))$; hence,

$$\sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{t} \gamma_j'^z(P(s')) \geq c(Y_i) - c(\overline{y}^q) + \sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{q} \gamma_j^z(P(s')).$$

By the construction of $(\gamma_i)_{i\in N}$,

$$c(Y_i) - c(\overline{y}^q) + \sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{q} \gamma_j^z(P(s')) > \sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{q} \gamma_j^z(P(s))$$

Thus,

$$\sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{t} \gamma_j'^z(P(s')) > \sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{q} \gamma_j^z(P(s)).$$

Note that $\sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{q} \gamma_j^z(P(s)) = \sum_{j\in P(s')\cap D\cap\mathcal{N}(0,\overline{y}^q]} \sum_{z=1}^{t} \gamma_j^z(P(s))$. Hence, there is $j \in P(s') \cap D \cap \mathcal{N}(0,\overline{y}^q]$ such that $\sum_{z=1}^{t} \gamma_j'^z(P(s')) > \sum_{z=1}^{t} \gamma_j^z(P(s))$, which implies that j is made worse off by this deviation.

In conclusion, in any case, if some member of D is made better off, then other members are made worse off. ||

By Claims 4, 5, and 6, no coalition can profitably deviate from s. ■

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